The Viscoelastic Seismic Model: Existence, Uniqueness and Differentiability with Respect to Parameters

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Abstract

In this thesis we prove the well-posedness of the viscoelastic wave equation for the generalized standard linear solid under given initial values and certain homogeneous boundary conditions. Also we show, that the solution of this equation is Fréchet-differentiable for the material parameters, the initial values and the external force density and stress rate. Finally, we derive the adjoint of the bounded linear operator given by this derivative at some parameter point.

A key concept in these proofs is the application of a variable transformation. It transforms the viscoelastic equation into an alternative form, which might be of interest in its own right. All results are formulated in terms of the original as well as the transformed variables.

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Contents

Chapter 1

Introduction

In this thesis we prove the unique solvability of the viscoelastic initial-boundary value problem

$$
\begin{aligned}
\rho(x)\partial_t \mathbf{v}(x,t) \;&=\; \operatorname{div}\boldsymbol{\sigma}(x,t) \;+\; \mathbf{f}(x,t)\,,\\
\partial_t \boldsymbol{\sigma}(x,t) \;&=\; C\Big(\mu_H(x) + \sum_{l=1}^{L}\mu_{M,l}(x)\;,\;\kappa_H(x) + \sum_{l=1}^{L}\kappa_{M,l}(x)\Big)\,\varepsilon(\mathbf{v})(x,t)\\
&\qquad\qquad + \sum_{l=1}^{L}\boldsymbol{\eta}_l(x,t) \;+\; \mathbf{g}(x,t)\,, \qquad\qquad (1.1)\\
\tau_{\boldsymbol{\sigma},l}(x)\partial_t \boldsymbol{\eta}_l(x,t) \;&=\; -C\big(\mu_{M,l}(x)\,,\,\kappa_{M,l}(x)\big)\,\varepsilon(\mathbf{v})(x,t) \;-\; \boldsymbol{\eta}_l(x,t)\,, \quad l = 1,\dots,L\,,
\end{aligned}
$$

$x \in D,\ t \in [0,t_1]$,

$$
\mathbf{v}(x,0) \;=\; \mathbf{v}^{(0)}(x)\,, \qquad \boldsymbol{\sigma}(x,0) \;=\; \boldsymbol{\sigma}^{(0)}(x)\,, \qquad \boldsymbol{\eta}_l(x,0) \;=\; \boldsymbol{\eta}_l^{(0)}(x)\,, \quad l = 1,\dots,L\,,
$$

$x \in D$,

$$
\mathbf{v}(x,t) \;=\; \mathbf{0}\,,\ x \in \partial D_D\,,\ t \in [0,t_1]\,, \qquad \mathbf{n}(x)^{\top}\boldsymbol{\sigma}(x,t) \;=\; \mathbf{0}\,,\ x \in \partial D_N\,,\ t \in [0,t_1]\,,
$$

with $\emptyset \neq D \subseteq \mathbb{R}^3$ open, $\partial D = \partial D_D \dot\cup \partial D_N$ and $\mathbf{n}$ denoting the outer unit normal vector, for the velocity $\mathbf{v}$, the stress $\boldsymbol{\sigma}$ and the memory variables $\boldsymbol{\eta}_l$ in a weak sense in space and a classical sense in time. The parameter functions will be explained in detail later. To achieve this goal we interpret (1.1) as the evolution equation

$$
\begin{aligned}
u'(t) \;&=\; -Au(t) \;+\; f(t)\,, \qquad\qquad t \in [0,t_1]\,,\\
u(0) \;&=\; u_0
\end{aligned}
\qquad (1.2)
$$

with a linear operator $A : X \supseteq D(A) \to X$ on a Hilbert space $\big(X, (\,.\,,\,.\,)_X\big)$. The crucial idea is the application of a variant of the Theorem of Hille-Yosida under the premise that A is maximal monotone with respect to a certain weighted scalar product $(.\,,.)_T$ on X. To determine this scalar product we perform the variable transformation

$$\begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma}_H \\ \boldsymbol{\sigma}_{M,1} \\ \vdots \\ \boldsymbol{\sigma}_{M,L} \end{pmatrix} \quad := \quad T \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma} \\ \boldsymbol{\eta}_1 \\ \vdots \\ \boldsymbol{\eta}_L \end{pmatrix} \quad := \quad \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma} + \sum_{l=1}^{L} \tau_{\boldsymbol{\sigma},l}\boldsymbol{\eta}_l \\ -\tau_{\boldsymbol{\sigma},1}\boldsymbol{\eta}_1 \\ \vdots \\ -\tau_{\boldsymbol{\sigma},L}\boldsymbol{\eta}_L \end{pmatrix},$$

which converts (1.1) into the initial-boundary value problem

$$\begin{aligned} \rho(x)\partial_t\mathbf{v}(x,t) &= \operatorname{div}\Big(\boldsymbol{\sigma}_H + \sum_{l=1}^{L}\boldsymbol{\sigma}_{M,l}\Big)(x,t) \,+\, \mathbf{f}(x,t)\,, \\ \partial_t\boldsymbol{\sigma}_H(x,t) &= C\big(\mu_H(x)\,,\,\kappa_H(x)\big)\,\varepsilon(\mathbf{v})(x,t) \,+\, \mathbf{g}(x,t)\,, \\ \partial_t\boldsymbol{\sigma}_{M,l}(x,t) &= C\big(\mu_{M,l}(x)\,,\,\kappa_{M,l}(x)\big)\,\varepsilon(\mathbf{v})(x,t) \,-\, \frac{1}{\tau_{\boldsymbol{\sigma},l}(x)}\,\boldsymbol{\sigma}_{M,l}(x,t)\,, \\ & \qquad l = 1,\ldots,L\,, \end{aligned} \tag{1.3}$$

$x \in D,\ t \in [0,t_1]$,

$$\mathbf{v}(x,0) \,=\, \mathbf{v}^{(0)}(x)\,, \quad \boldsymbol{\sigma}_H(x,0) \,=\, \boldsymbol{\sigma}_H^{(0)}(x)\,, \quad \boldsymbol{\sigma}_{M,l}(x,0) \,=\, \boldsymbol{\sigma}_{M,l}^{(0)}(x)\,,\ l = 1,\ldots,L\,,$$

$x \in D$,

$$\mathbf{v}(x,t) \,=\, \mathbf{0}\,, \quad x \in \partial D_D\,, \qquad \mathbf{n}(x)^\top\Big(\boldsymbol{\sigma}_H + \sum_{l=1}^{L}\boldsymbol{\sigma}_{M,l}\Big)(x,t) \,=\, \mathbf{0}\,, \quad x \in \partial D_N\,,$$

$t \in [0,t_1]$.

From an abstract, functional analytic point of view this variable transformation can be considered as the application of the bounded and boundedly invertible linear operator $T \in \mathcal{L}(X)$ which converts the evolution equation (1.2) into the evolution equation

$$\begin{aligned} w'(t) &= -Bw(t) \,+\, Tf(t)\,, \qquad t \in [0,t_1]\,, \\ w(0) &= Tu_0 \end{aligned}$$

on the basis of the linear operator $B : X \supseteq \mathcal{D}(B) \to X$ with $B = TAT^{-1}$. System (1.3) in turn can easily physically be interpreted as a model for wave propagation in the standard linear solid. Also the form of the corresponding physical energy scalar product $(.\,,.)_E$ becomes evident.

The operator B is of the form $-P_1Q + P_2$ where $P_1, P_2 \in \mathcal{L}(X)$ are bounded linear operators on X which depend on the material parameters of the medium, whereas $Q : X \supseteq \mathcal{D}(B) \to X$ is an unbounded linear operator which is independent of the material parameters. In addition P_1 is invertible. It turns out that $(.\,,.)_E = (P_1^{-1}.\,,.)_X$ and B is maximal monotone with respect to $(.\,,.)_E$. On the other hand, the appropriate energy scalar product $(.\,,.)_T$ with respect to which A is a maximal monotone operator, is given by back transforming $(.\,,.)_E$ as $(.\,,.)_T = (T\cdot\,, T\cdot)_E$.

Working with these two scalar products, that is the scalar product $(.\,,.)_T$ in connection with the operator A and the scalar product $(.\,,.)_E$ in connection with the operator B, has several benefits. Besides the property of turning A and B into maximal monotone operators and therefore serving as the basis of our existence and uniqueness result, it becomes also possible to quantify the exact energy loss over time.

In a second part of this thesis we show that the solution $\big(\mathbf{v}, \boldsymbol{\sigma}, (\boldsymbol{\eta}_l)_l\big)^\top$ of equation (1.1) is a Fréchet-differentiable function of its material parameters, initial value und right-hand side. The same applies for the solution $\big(\mathbf{v}, \boldsymbol{\sigma}_H, (\boldsymbol{\sigma}_{M,l})_l\big)^\top$ of (1.3).

For gradient based methods used in seismic imaging (see e.g. [17], [18] and [19]) also the adjoint Fréchet derivative at any parameter point is needed, which we derive, too. For its computation we need the adjoint operator A^* of A or B^* of B. As a further advantage of the use of $(.\,,.)_T$ and $(.\,,.)_E$ it turns out that P_1Q is skew symmetric and P_2 is symmetric with respect to $(.\,,.)_E$. In the same way A is the sum of a skew-symmetric and a symmetric operator with respect to $(.\,,.)_T$ since $A = -T^{-1}P_1QT + T^{-1}P_2T$. Thus the adjoint operators only differ by some minus signs from the original ones and the same numerical implementation can be used to evaluate both of them.

The fundamental theoretical basis for this thesis is [14]. Concerning the application of semigroups and maximal monotone operators to prove existence and uniqueness of the solution of the evolution equation in question and also the methods to prove differentiability for parameters in the abstract setting of evolution equations, this thesis can be considered as an application and slide variation of the ideas and methods presented there.

Chapter 2

Viscoelasticity: The Model

In this section we list some common forms of viscoelastic equations and also explain how they are related to each other. All calculations are done in an informal way without mathematical rigor.

2.1 Formulation as an Integro-Differential Equation

To describe viscoelastic wave propagation in a material whose parameters do not depend on time, [20] states the following equation. We use a slightly different notation:

$$\rho(x)\frac{\partial^2 u_i}{\partial t^2}(x,t) \;=\; \sum_{r=1}^{3} \frac{\partial \sigma_{ir}}{\partial x_r}(x,t) \;+\; f_i(x,t)\,, \tag{2.1}$$

$$\sigma_{ij}(x,t) \;=\; \int_{-\infty}^{\infty} \sum_{r,s=1}^{3} \Psi_{ijrs}(x,t-t')\frac{\partial u_r}{\partial x_s}(x,t')\,dt' \;+\; \tilde{g}_{ij}(x,t)\,, \tag{2.2}$$

$i,j = 1,2,3$.

Here, the space variable $x = (x_1, x_2, x_3)$ can be assumed to range over a subset D of $\mathbb{R}^3$ and the time variable t is assumed to be an element of the real axis $\mathbb{R}$.

The solution of this equation consists of the pair $(\mathbf{u}, \boldsymbol{\sigma})$, where $\mathbf{u} = (u_i)_{i=1,2,3} : D \times \mathbb{R} \to \mathbb{R}^3$ denotes the displacement vector and $\boldsymbol{\sigma} = (\sigma_{ij})_{i,j=1,2,3} : D \times \mathbb{R} \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ is the stress tensor.

As material parameters the equation contains the mass density $\rho : D \to \mathbb{R}$ and the rate-of-relaxation function $\boldsymbol{\Psi} = (\Psi_{ijrs})_{i,j,r,s=1,2,3} : D \times \mathbb{R} \to \mathbb{R}^{3\times 3\times 3\times 3}$. Instead of a function, $\boldsymbol{\Psi}$ can also be a distribution, in which case the integral in (2.2) has to be formally interpreted as a convolution. Restrictions on $\boldsymbol{\Psi}$ are causality, that is $\mathbf{\Psi}(\cdot, t) = 0$ for $t < 0$, and the symmetries

$$\Psi_{ijrs} \;=\; \Psi_{jirs} \;=\; \Psi_{rsij} \;=\; \Psi_{ijsr} \;=\; \Psi_{jirs}\,,$$

$i, j, r, s = 1, 2, 3$, where the last two equalities are a consequence of the first two.

Finally, (2.1) and (2.2) contain the external volume force density $\mathbf{f} = (f_i)_{i=1,2,3} : D \times \mathbb{R} \to \mathbb{R}^3$ and external stress $\tilde{\mathbf{g}} = (\tilde{g}_{ij})_{i,j=1,2,3} : D \times \mathbb{R} \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$.

2.2 Formulation for the Generalized Standard Linear Solid

As a special case of system (2.1), (2.2) we derive the equation system describing the generalized standard linear solid:

$$\rho(x)\frac{\partial v_i}{\partial t}(x,t) \;=\; \sum_{r=1}^{3} \frac{\partial \sigma_{ir}}{\partial x_r}(x,t) \;+\; f_i(x,t)\,, \tag{2.3}$$

$$\frac{\partial \sigma_{ij}}{\partial t}(x,t) \;=\; \sum_{r,s=1}^{3} a_{ijrs}(x)\frac{\partial v_r}{\partial x_s}(x,t) \;+\; \sum_{k=1}^{L} \eta_{kij}(x,t) \;+\; g_{ij}(x,t)\,, \tag{2.4}$$

$$\frac{\partial \eta_{lij}}{\partial t}(x,t) \;=\; -\,\frac{1}{\tau_{\boldsymbol{\sigma},l}(x)}\Big(\sum_{r,s=1}^{3} c_{lijrs}(x)\frac{\partial v_r}{\partial x_s}(x,t) \;+\; \eta_{lij}(x,t)\Big)\,, \tag{2.5}$$

with the external stress rate $\mathbf{g} = (g_{ij})_{i,j=1,2,3} := \partial\tilde{\mathbf{g}}/\partial t$ and $l = 1, \ldots, L$ with an $L \in \mathbb{N}$. It is explained and used in [4], [1], [6], [12].

Its solution consists in the tuple $\big(\mathbf{v}, \boldsymbol{\sigma}, (\boldsymbol{\eta}_l)_{l=1,\ldots,L}\big)$, where $(v_i)_{i=1,2,3} = \mathbf{v} := \partial\mathbf{u}/\partial t$ is the time derivative of the displacement vector, also called "velocity", and $\boldsymbol{\eta}_l = (\eta_{lij})_{i,j=1,2,3} : D \times \mathbb{R} \to \mathbb{R}^{3\times 3}_{\mathrm{sym}},\ \ l = 1, \ldots, L$ are so called memory tensors.

The parameters $\boldsymbol{a} = (a_{ijrs})_{i,j,r,s=1,2,3},\ \ \boldsymbol{c}_l = (c_{lijrs})_{i,j,r,s=1,2,3} : D \to \mathbb{R}^{3\times 3\times 3\times 3}$ are elastic stiffness tensors and the parameters $\tau_{\boldsymbol{\sigma},l} : D \to \mathbb{R}$ are so called stress relaxation times for every index l respectively.

To derive (2.3)–(2.5) from (2.1), (2.2) in an informal way, we choose

$$\Psi_{ijrs}(x,t) \;=\; a_{ijrs}(x)\delta(t) \;+\; \sum_{l=1}^{L} b_{lijrs}(x,t)H(t) \tag{2.6}$$

with

$$b_{lijrs}(x,t) \;:=\; -\frac{c_{lijrs}(x)}{\tau_{\boldsymbol{\sigma},l}(x)}\, e^{-t/\tau_{\boldsymbol{\sigma},l}(x)}\,.$$

In (2.6) (by an abuse of notation) $\delta(t)$ stands for the delta distribution on the real axis.

Equation (2.3) and equation (2.1) only differ in the notation on the left-hand side. To get (2.4), we plug (2.6) and $v_r = \partial u_r / \partial t$ into the time derivative of (2.2) like

$$\begin{aligned}
\frac{\partial \sigma_{ij}}{\partial t}(x,t) &= \int_{-\infty}^{\infty} \sum_{r,s=1}^{3} \frac{\partial \Psi_{ijrs}}{\partial t}(x,t-t') \frac{\partial u_r}{\partial x_s}(x,t')\,dt' + \frac{\partial \tilde{g}_{ij}}{\partial t}(x,t) \\
&= \int_{-\infty}^{\infty} \sum_{r,s=1}^{3} \Psi_{ijrs}(x,t-t') \frac{\partial v_r}{\partial x_s}(x,t')\,dt' + g_{ij}(x,t) \\
&= \sum_{r,s=1}^{3} a_{ijrs}(x) \frac{\partial v_r}{\partial x_s}(x,t) + \sum_{l=1}^{L} \eta_{lij}(x,t) + g_{ij}(x,t)\,,
\end{aligned}$$

where we define

$$\eta_{lij}(x,t) := \int_{-\infty}^{t} \sum_{r,s=1}^{3} b_{lijrs}(x,t-t') \frac{\partial v_r}{\partial x_s}(x,t')\,dt'\,.$$

And finally, η_{lij} solves the initial value problem:

$$\begin{aligned}
\frac{\partial \eta_{lij}}{\partial t}(x,t) &= -\frac{1}{\tau_{\boldsymbol{\sigma},l}(x)} \Big(\sum_{r,s=1}^{3} c_{lijrs}(x) \frac{\partial v_r}{\partial x_s}(x,t) + \eta_{lij}(x,t) \Big)\,, \\
\eta(x,0) &= \eta_{lij,0}(x) := \int_{-\infty}^{0} \sum_{r,s=1}^{3} b_{lijrs}(x,-t') \frac{\partial v_r}{\partial x_s}(x,t')\,dt'\,,
\end{aligned}$$

which can by proven by a direct calculation or via the intermediate step

$$\eta_{lij}(x,t) = e^{-t/\tau_{\boldsymbol{\sigma},l}(x)} \eta_{lij,0}(x) - \sum_{r,s=1}^{3} \frac{c_{lijrs}(x)}{\tau_{\boldsymbol{\sigma},l}(x)} \int_{0}^{t} e^{-(t-t')/\tau_{\boldsymbol{\sigma},l}(x)} \frac{\partial v_r}{\partial x_s}(x,t')\,dt'\,.$$

So equation (2.5) holds, too.

2.3 Formulation for Isotropic Materials

In the isotropic case one can further simplify the elastic stiffness tensors $\boldsymbol{a}$ and $\boldsymbol{c}_l$ by introducing pairs of Lamé-parameters $\tilde{\mu}_{\boldsymbol{a}}$, $\lambda_{\boldsymbol{a}}$ and $\tilde{\mu}_{\boldsymbol{c},l}$, $\lambda_{\boldsymbol{c},l}$, respectively, which are functions $D \to \mathbb{R}$ such that $\tilde{\mu}_{\boldsymbol{a}}, \tilde{\mu}_{\boldsymbol{c},l} > 0$, $\lambda_{\boldsymbol{a}} > -(2/3)\tilde{\mu}_{\boldsymbol{a}}$ and $\lambda_{\boldsymbol{c},l} > -(2/3)\tilde{\mu}_{\boldsymbol{c},l}$, $l = 1, \ldots, L$. Then

$$a_{ijrs}(x) = \lambda_{\boldsymbol{a}}(x)\delta_{ij}\delta_{rs} + \tilde{\mu}_{\boldsymbol{a}}(x)\big(\delta_{ir}\delta_{js} + \delta_{jr}\delta_{is}\big)$$

and

$$c_{lijrs}(x) = \lambda_{\boldsymbol{c},l}(x)\delta_{ij}\delta_{rs} + \tilde{\mu}_{\boldsymbol{c},l}(x)\big(\delta_{ir}\delta_{js} + \delta_{jr}\delta_{is}\big)\,,$$

$l = 1, \dots, L$, where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ otherwise, is the Kronecker-symbol.

In three space dimensions Lamé's First Parameter $\lambda_{\boldsymbol{a}}$ and $\lambda_{\boldsymbol{c},l}$ can be expressed as $\lambda_{\boldsymbol{a}} = \tilde{\kappa}_{\boldsymbol{a}} - (2/3)\tilde{\mu}_{\boldsymbol{a}}$ and $\lambda_{\boldsymbol{c},l} = \tilde{\kappa}_{\boldsymbol{c},l} - (2/3)\tilde{\mu}_{\boldsymbol{c},l}$ by the shear modulus $\tilde{\mu}_{\boldsymbol{a}}$ and $\tilde{\mu}_{\boldsymbol{c},l}$ and the bulk modulus $\tilde{\kappa}_{\boldsymbol{a}}$ and $\tilde{\kappa}_{\boldsymbol{c},l}$, respectively (see [13] for example).

2.4 Formulation Using Scaling Variables

By additionally introducing scaling variables τ_{S} and τ_{P} for the shear and bulk modulus respectively like done in [2] and setting $\tilde{\mu}_a = \tilde{\mu}(1 + L\tau_{\mathrm{S}})$, $\lambda_a = \tilde{\kappa}(1 + L\tau_{\mathrm{P}}) - (2/3)\tilde{\mu}(1 + L\tau_{\mathrm{S}})$ and $\tilde{\mu}_{\boldsymbol{c},l} = \tilde{\mu}\tau_{\mathrm{S}}$, $\lambda_{\boldsymbol{c},l} = \tilde{\kappa}\tau_{\mathrm{P}} - (2/3)\tilde{\mu}\tau_{\mathrm{S}}$ for $l \in \{1, \dots, L\}$ we arrive at the equation which has been the starting point for this thesis:

$$\begin{aligned}
\rho\partial_t \mathbf{v} &= \operatorname{div}\boldsymbol{\sigma} \;+\; \mathbf{f}\,, \\
\partial_t \boldsymbol{\sigma} &= 2\tilde{\mu}(1 + L\tau_{\mathrm{S}})\operatorname{dev}\varepsilon(\mathbf{v}) \;+\; \tilde{\kappa}(1 + L\tau_{\mathrm{P}})\operatorname{div}\mathbf{v}\mathbf{I} \;+\; \sum_{l=1}^{L}\boldsymbol{\eta}_l \;+\; \mathbf{g}\,, \qquad (2.7) \\
\tau_{\boldsymbol{\sigma},l}\partial_t\boldsymbol{\eta}_l &= -2\tilde{\mu}\tau_{\mathrm{S}}\operatorname{dev}\varepsilon(\mathbf{v}) \;-\; \tilde{\kappa}\tau_{\mathrm{P}}\operatorname{div}\mathbf{v}\mathbf{I} \;-\; \boldsymbol{\eta}_l\,, \qquad l = 1, \dots, L\,.
\end{aligned}$$

Here and in the sequel, $\mathbf{I}$ is the constant unit matrix in $\mathbb{R}^{3\times 3}$. The differential operators div and ε on the right hand side of (2.7) are defined as follows:

$$\begin{aligned}
\operatorname{div}\mathbf{v}(x,t) &= \sum_{i=1}^{3}\partial_{x_i}v_i(x,t)\,, \quad \text{where } \mathbf{v}(x,t) = (v_i(x,t))_{i=1,2,3}\,, \\
\operatorname{div}\boldsymbol{\sigma}(x,t) &= \sum_{j=1}^{3}\partial_{x_j}\boldsymbol{\sigma}_{*j}(x,t) \quad \text{with } \boldsymbol{\sigma}_{*j}(x,t) = (\sigma_{ij}(x,t))_{i=1,2,3}\,,\ j = 1,2,3\,, \\
\varepsilon(\mathbf{v})(x,t) &= \frac{1}{2}\big(D_x\mathbf{v}(x,t) + (D_x\mathbf{v}(x,t))^\top\big) \text{ with the Jacobian } D_x\mathbf{v} \text{ w.r.t. } x\,.
\end{aligned}$$

For any Matrix $\mathbf{M} \in \mathbb{R}^{3\times 3}$, the deviatoric part $\operatorname{dev}\mathbf{M}$ is defined as

$$\operatorname{dev}\mathbf{M} = \mathbf{M} - \frac{1}{3}\operatorname{trace}(\mathbf{M})\mathbf{I}\,. \qquad (2.8)$$

2.5 Formulation Used in This Thesis

For the formulation for isotropic materials used in this thesis we introduce the linear maps

$$C(m,k):\ \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}\,, \qquad C(m,k)\mathbf{M} = m\mathbf{M} + \frac{k-m}{3}\operatorname{trace}(\mathbf{M})\mathbf{I} \qquad (2.9)$$

depending on two real parameters m and k. So

$$C(m,k)\mathbf{M} \quad = \quad m \operatorname{dev} \mathbf{M} \,+\, k \, \frac{\operatorname{trace}(\mathbf{M})}{3} \, \mathbf{I} \,.$$

Note that $C(m,k)$ maps $\mathbb{R}^{3\times 3}_{\mathrm{sym}}$ into $\mathbb{R}^{3\times 3}_{\mathrm{sym}}$.

Using this notation and the definition $\tilde{\mathbf{f}} \coloneqq \mathbf{f}/\rho$ and taking into account that $\operatorname{trace}\big(\varepsilon(\mathbf{v})\big) = \operatorname{div}\mathbf{v}$, we can write equation (2.7) in the form

$$\begin{aligned}
\partial_t \mathbf{v} \quad &= \quad \frac{1}{\rho} \operatorname{div} \boldsymbol{\sigma} \,+\, \tilde{\mathbf{f}} \,, \\
\partial_t \boldsymbol{\sigma} \quad &= \quad C\Big(2\tilde{\mu}(1+L\tau_{\mathrm{S}})\,,\, 3\tilde{\kappa}(1+L\tau_{\mathrm{P}})\Big)\varepsilon(\mathbf{v}) \,+\, \sum_{l=1}^{L} \boldsymbol{\eta}_l \,+\, \mathbf{g}\,, \qquad (2.10) \\
\partial_t \boldsymbol{\eta}_l \quad &= \quad -\frac{1}{\tau_{\boldsymbol{\sigma},l}} C\Big(2\tilde{\mu}\tau_{\mathrm{S}}\,,\, 3\tilde{\kappa}\tau_{\mathrm{P}}\Big)\varepsilon(\mathbf{v}) \,-\, \frac{1}{\tau_{\boldsymbol{\sigma},l}} \boldsymbol{\eta}_l\,, \qquad l = 1,\dots,L\,.
\end{aligned}$$

After rescaling $\tilde{\mu}$ and $\tilde{\kappa}$ to

$$\mu \quad \coloneqq \quad 2\tilde{\mu}\,, \qquad\qquad \kappa \quad \coloneqq \quad 3\tilde{\kappa} \qquad (2.11)$$

equation (2.10) takes the form

$$\begin{aligned}
\partial_t \mathbf{v} \quad &= \quad \frac{1}{\rho} \operatorname{div} \boldsymbol{\sigma} \,+\, \tilde{\mathbf{f}} \,, \\
\partial_t \boldsymbol{\sigma} \quad &= \quad C\Big(\mu(1+L\tau_{\mathrm{S}})\,,\, \kappa(1+L\tau_{\mathrm{P}})\Big)\varepsilon(\mathbf{v}) \,+\, \sum_{l=1}^{L} \boldsymbol{\eta}_l \,+\, \mathbf{g}\,, \qquad (2.12) \\
\partial_t \boldsymbol{\eta}_l \quad &= \quad -\frac{1}{\tau_{\boldsymbol{\sigma},l}} C\Big(\mu\tau_{\mathrm{S}}\,,\, \kappa\tau_{\mathrm{P}}\Big)\varepsilon(\mathbf{v}) \,-\, \frac{1}{\tau_{\boldsymbol{\sigma},l}} \boldsymbol{\eta}_l\,, \qquad l = 1,\dots,L\,.
\end{aligned}$$

Yet, we admit more degrees of freedom in the parameters than (2.7) by allowing $\mu\tau_{\mathrm{S}}$ and $\kappa\tau_{\mathrm{P}}$ to depend on $l \in \{1,\dots,L\}$. Accordingly we denote them by $\mu_{M,l}$ and $\kappa_{M,l}$ respectively and replace $L\mu\tau_{\mathrm{S}}$ by $\sum_{l=1}^{L}\mu_{M,l}$ and $L\kappa\tau_{\mathrm{P}}$ by $\sum_{l=1}^{L}\kappa_{M,l}$. Furthermore we define

$$\mu_H \,\coloneqq\, \mu\,, \qquad \kappa_H \,\coloneqq\, \kappa\,, \qquad \vartheta \,\coloneqq\, \frac{1}{\rho}\,, \qquad \omega_{\boldsymbol{\sigma},l} \,\coloneqq\, \frac{1}{\tau_{\boldsymbol{\sigma},l}}\,, \quad l = 1,\dots,L\,.$$

Then (2.12) reads

$$\begin{aligned}
\partial_t \mathbf{v} \quad &= \quad \vartheta \operatorname{div} \boldsymbol{\sigma} \,+\, \tilde{\mathbf{f}} \,, \\
\partial_t \boldsymbol{\sigma} \quad &= \quad C\left(\mu_H + \sum_{l=1}^{L}\mu_{M,l}\,,\, \kappa_H + \sum_{l=1}^{L}\kappa_{M,l}\right)\varepsilon(\mathbf{v}) \,+\, \sum_{l=1}^{L} \boldsymbol{\eta}_l \,+\, \mathbf{g}\,, \qquad (2.13) \\
\partial_t \boldsymbol{\eta}_l \quad &= \quad -\omega_{\boldsymbol{\sigma},l} C\big(\mu_{M,l}\,,\, \kappa_{M,l}\big)\,\varepsilon(\mathbf{v}) \,-\, \omega_{\boldsymbol{\sigma},l}\boldsymbol{\eta}_l\,, \qquad l = 1,\dots,L\,.
\end{aligned}$$

For further use we abbreviate

$$\boldsymbol{\eta} \coloneqq (\boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_L)^\top .$$

As before the solution $(\mathbf{v}, \boldsymbol{\sigma}, \boldsymbol{\eta})^\top$ of (2.13) depends on the space variable $x = (x_1, x_2, x_3) \in D \subseteq \mathbb{R}^3$ as well as the time variable $t \in \mathbb{R}$ which we restrict to $t \in [0, t_1]$ with some $t_1 > 0$ from now on. Also the inhomogeneity consisting of $\tilde{\mathbf{f}}$ and $\mathbf{g}$ depends on x and t, whereas the material parameters ϑ, μ_H, $\mu_{M,l}$, κ_H, $\kappa_{M,l}$, $\omega_{\boldsymbol{\sigma},l}$ only depend on x.

2.6 Formulation as an Evolution Equation

To treat equation (2.13) in a mathematically sound way we formulate it as an evolution equation. This means we will interpret the solution $(x, t) \mapsto (\mathbf{v}, \boldsymbol{\sigma}, \boldsymbol{\eta})^\top(x, t)$ as a function from the time interval $[0, t_1]$ into a Hilbert space X, which maps a point t in time to the function $(\mathbf{v}, \boldsymbol{\sigma}, \boldsymbol{\eta})^\top(\cdot, t)$ in the space variable $x \in D$. By an abuse of notation we henceforth denote this function itself by $(\mathbf{v}, \boldsymbol{\sigma}, \boldsymbol{\eta})^\top(t)$.

For every point t in time the right-hand side of (2.13) can be seen as the sum of the negative image of $(\mathbf{v}, \boldsymbol{\sigma}, \boldsymbol{\eta})^\top(t)$ under a linear operator A on X and the inhomogeneity $f(t) \coloneqq \big(\tilde{\mathbf{f}}(t), \mathbf{g}(t), \mathbf{0}\big)^\top$. The operator A then has the form

$$A \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma} \\ \boldsymbol{\eta}_1 \\ \vdots \\ \boldsymbol{\eta}_L \end{pmatrix} = - \begin{pmatrix} \vartheta \operatorname{div} \boldsymbol{\sigma} \\ C\Big(\mu_H + \sum_{l=1}^{L} \mu_{M,l} \, , \, \kappa_H + \sum_{l=1}^{L} \kappa_{M,l}\Big)\, \varepsilon(\mathbf{v}) \, + \, \sum_{l=1}^{L} \boldsymbol{\eta}_l \\ -\omega_{\boldsymbol{\sigma},1} C\big(\mu_{M,1} \, , \, \kappa_{M,1}\big)\, \varepsilon(\mathbf{v}) \, - \, \omega_{\boldsymbol{\sigma},1} \boldsymbol{\eta}_1 \\ \vdots \\ -\omega_{\boldsymbol{\sigma},L} C\big(\mu_{M,L} \, , \, \kappa_{M,L}\big)\, \varepsilon(\mathbf{v}) \, - \, \omega_{\boldsymbol{\sigma},L} \boldsymbol{\eta}_L \end{pmatrix} . \tag{2.14}$$

By $'$ we denote the derivative of a function of one variable. With the notation $u(t) \coloneqq (\mathbf{v}, \boldsymbol{\sigma}, \boldsymbol{\eta})^\top(t)$ and by adding an initial-value $u_0 = (\mathbf{v}^{(0)}, \boldsymbol{\sigma}^{(0)}, \boldsymbol{\eta}^{(0)})^\top$ which is to be understood as a function in the space variable x, equation (2.13) takes on the form of an evolution equation:

$$\begin{aligned} u'(t) &= -Au(t) + f(t) , \qquad && t \in [0, t_1] , \\ u(0) &= u_0 . \end{aligned} \tag{2.15}$$

The following two sections recall some basic facts about the solvability of such an equation.

Chapter 3

Basic Facts on Abstract Evolution Equations

Throughout this chapter we assume $\big(X, (.\,,.)\big)$ to be a general real Hilbert space, $\|.\|$ the norm induced by the scalar product $(.\,,.)$, $R : X \supseteq \mathcal{D}(R) \to X$ a linear but not necessarily bounded operator with domain of definition $\mathcal{D}(R)$ and $t_1 > 0$ a real constant. By $'$ we denote the derivative of any function $v : I \to X$ on a real interval $I \subseteq \mathbb{R}$ with values in X.

The purpose of this section is to collect some facts on abstract evolution equations of the form

$$\begin{aligned} v'(t) &= -Rv(t) + f(t)\,, \qquad\qquad t \in [0, t_1]\,, \\ v(0) &= v_0 \end{aligned} \tag{3.1}$$

for the special case of R being maximal monotone (see Definition 1) and v_0 and f having several degrees of regularity. At least we assume $v_0 \in X$ and $f \in L^1\big((0, t_1), X\big)$. In particular we prove existence, uniqueness and regularity of a function $v : [0, t_1] \to X$ solving (3.1). These results are needed in later chapters and are especially presented to serve those purposes.

3.1 The Homogeneous Equation

To prove existence, uniqueness and stability of the solution of the homogeneous evolution problem

$$\begin{aligned} v'(t) &= -Rv(t)\,, \qquad\qquad t \in [0, \infty)\,, \\ v(0) &= v_0 \end{aligned} \tag{3.2}$$

on the unrestricted interval $[0, \infty)$ we are going to make use of Theorem 3 and Theorem 4. Both statements are formulated and proven in [5] as Theorem 7.4 (Hille-Yosida). They are based on the following definition.

Definition 1. *A linear operator $R : X \supseteq \mathcal{D}(R) \to X$ is called* ***monotone****, if*

$$(Rv, v) \quad \geq \quad 0\,, \qquad\qquad v \in \mathcal{D}(R)\,. \tag{3.3}$$

If in addition

$$\mathcal{R}(\mathrm{Id} + R) \quad = \quad X\,, \tag{3.4}$$

where Id *denotes the identity map on* $\mathcal{D}(R)$ *and* Id$+R$ *is considered as an operator from* $\mathcal{D}(R)$ *to* X*, it is called* ***maximal monotone****.* □

Lemma 2. *For a maximal monotone operator $R : X \supseteq \mathcal{D}(R) \to X$ it holds*

(a) $\mathcal{D}(R)$ is dense in X.

(b) R is closed.

(c) For every $\alpha > 0$, the operator $\mathrm{Id} + \alpha R :\ \mathcal{D}(R) \to X$ is bijective. Moreover $(\mathrm{Id} + \alpha R)^{-1} : \big(X, \|.\|\big) \to \big(\mathcal{D}(R), \|.\|\big)$ is bounded with $\big\|(\mathrm{Id} + \alpha R)^{-1}\big\| \leq 1$.

Proof. This is proven in [5] as Proposition 7.1. □

Theorem 3. *Let $R : X \supseteq \mathcal{D}(R) \to X$ be a maximal monotone linear operator and $v_0 \in \mathcal{D}(R)$. Then there exists a unique $v \in C^1\big([0,\infty), X\big) \cap C\big([0,\infty), \mathcal{D}(R)\big)$, which solves (3.2). Here, $\mathcal{D}(R)$ is equipped with the graph norm $\|x\|_R := \|x\| + \|Rx\|$, $x \in \mathcal{D}(R)$.*

Proof. For a proof we refer to [5], Theorem 7.4. □

Theorem 4. *Let $R : X \supseteq \mathcal{D}(R) \to X$ be maximal monotone, $v_0 \in \mathcal{D}(R)$, $I = [0,\infty)$ or $I = [0, t_1]$, and let $v \in C^1(I, X) \cap C\big(I, \mathcal{D}(R)\big)$ be a solution of*

$$\begin{aligned} v'(t) &= -Rv(t)\,, \qquad\qquad t \in I\,, \\ v(0) &= v_0\,. \end{aligned}$$

Then

$$\frac{d}{dt}\|v(t)\|^2 \quad = \quad -2\,\big(Rv(t), v(t)\big) \quad \leq \quad 0\,, \qquad\qquad t \in I\,. \tag{3.5}$$

In particular it holds

$$\|v(t)\| \quad \leq \quad \|v_0\|\,, \qquad\qquad t \in I\,. \tag{3.6}$$

Proof. The simple proof is part of the proof of Theorem 3 in [5]:

$$\frac{d}{dt}\big\|v(t)\big\|^2 \quad = \quad 2\,\big(v'(t)\,,\, v(t)\big) \quad = \quad -2\,\big(Rv(t)\,,\, v(t)\big) \quad \leq \quad 0\,, \qquad\qquad t \in I\,,$$

since R is monotone. So $t \mapsto \big\|v(t)\big\|$ is monotonically non-increasing on I, from which (3.6) follows. □

Theorem 3 guarantees the existence of a unique solution of (3.2) on the interval $[0,\infty)$. Such a solution can be restricted to yield a solution of (3.2) on the restricted interval $[0,t_1]$, where we chose $t_1>0$ arbitrarily. That even on $[0,t_1]$ the solution of (3.2) is unique, is the content of the following lemma.

Lemma 5. *Let $R : X \supseteq \mathcal{D}(R) \to X$ be maximal monotone, $v_0 \in \mathcal{D}(R)$, and let $v, \tilde{v} \in C^1\big([0,t_1],X\big) \cap C\big([0,t_1],\mathcal{D}(R)\big)$ be solutions of*

$$\begin{aligned} v'(t) &= -Rv(t)\,, \qquad\qquad t\in[0,t_1]\,,\\ v(0) &= v_0 \end{aligned} \tag{3.7}$$

Then $v=\tilde{v}$.

Proof. As $v, \tilde{v} \in C^1\big([0,t_1],X\big) \cap C\big([0,t_1],\mathcal{D}(R)\big)$ are solutions of (3.7) and R and the derivative are linear, the function $v-\tilde{v} \in C^1\big([0,t_1],X\big) \cap C\big([0,t_1],\mathcal{D}(R)\big)$ is a solution of (3.7) with $v_0=0$. Thus by (3.6) we have $\|v(t)-\tilde{v}(t)\|=0$, $t\in[0,t_1]$. So $v=\tilde{v}$. □

In the sequel the following definition will play a central role. It is taken from [10] and [5].

Definition 6. *A family of bounded linear operators $\big(S(t)\big)_{t\in[0,\infty)}$ on X which satisfies*

$$\begin{aligned} S(s+t) &= S(s)S(t)\,, \qquad\qquad s,t\in[0,\infty)\,,\\ S(0) &= \mathrm{Id}\,,\\ \lim_{s\to t} S(s)v_0 &= S(t)v_0\,, \qquad\qquad t\in[0,\infty)\,,\ v_0\in X\,, \end{aligned} \tag{3.8}$$

*is called a **strongly continuous semigroup** or C_0-**semigroup**. If $S(\cdot)$ is defined on all of $\mathbb{R}$ and (3.8) is satisfied for all $s,t\in\mathbb{R}$, it is called a **strongly continuous group** or C_0-**group**.*

If a C_0-semigroup $\big(S(t)\big)_{t\in[0,\infty)}$ satisfies

$$\big\|S(t)v_0\big\| \le \|v_0\|\,, \qquad t\in[0,\infty)\,, \qquad v_0\in X\,,$$

*it is called a **contraction semigroup**.* □

Definition 7. *For a maximal monotone operator $R : X \supseteq \mathcal{D}(R) \to X$ we define the family of linear operators $\big(S_R(t)\big)_{t\in[0,\infty)}$ by*

$$S_R(t):\ \big(\mathcal{D}(R),\|.\|\big) \to \big(X,\|.\|\big)\,, \qquad v_0 \mapsto v(t) \tag{3.9}$$

for every $t \in [0, \infty)$, where v is the unique solution of the evolution equation

$$\begin{aligned} v'(t) &= -Rv(t)\,, \qquad && t \in [0,\infty)\,, \\ v(0) &= v_0\,, \end{aligned} \tag{3.10}$$

in the space $C^1\big([0,\infty), X\big) \cap C\big([0,\infty), \mathcal{D}(R)\big)$, which by Theorem 3 exists. Due to (3.6)*, each $S_R(t)$ is bounded with $\|S_R(t)\| \le 1$, and according to Lemma 2*(a) *the space $\mathcal{D}(R)$ is dense in X. Therefore we can continuously extend $S_R(t)$ to*

$$S_R(t):\ \big(X, \|.\|\big) \to \big(X, \|.\|\big)\,.$$

$t \in [0, \infty)$. □

Lemma 8. *For a maximal monotone operator $R : X \supseteq \mathcal{D}(R) \to X$, the family $\big(S_R(t)\big)_{t\in[0,\infty)}$ introduced in Definition 7 forms a contraction semigroup. In particular it is*

$$\big\|S_R(t)v_0\big\| \ \le\ \|v_0\|\,, \qquad\qquad v_0 \in X\,, \quad t \in [0,\infty)\,. \tag{3.11}$$

Proof. In a first step let $v_0 \in \mathcal{D}(R)$. Then $S_R(0)v_0 = v_0$ follows from (3.10) and $\lim_{s\to t} S_R(s)v_0 = S_R(t)v_0$, $t \in [0,\infty)$ follows from $S_R(\cdot)v_0 \in C^1\big([0,\infty), X\big) \cap C\big([0,\infty), \mathcal{D}(R)\big)$ by definition. Also, for fixed $s \in [0,\infty)$, both $S_R(\cdot)S_R(s)v_0$ and $S_R(\,\cdot + s)v_0$ solve $v' = -Rv$, $v(0) = S_R(s)v_0$. So they are equal. Finally we recall (3.6) from which $\|S_R(t)v_0\| \le \|v_0\|$, $t \in [0,\infty)$ follows.

Now let $v_0 \in X$. Since $\mathcal{D}(R)$ is dense in X there is $(v_n)_{n\in\mathbb{N}}$ in $\mathcal{D}(R)$ with $\lim_{n\to\infty} v_n = v_0$. For the family $\big(S_R(t)\big)_{t\in[0,\infty)}$ of continuous extensions to X it follows

$$S_R(0)v_0 \ =\ \lim_{n\to\infty} S_R(0)v_n \ =\ \lim_{n\to\infty} v_n \ =\ v_0\,.$$

In the same way we prove $S_R(s+t)v_0 = S_R(s)S_R(t)v_0$, $s, t \in [0,\infty)$ and (3.11).

Finally we need to show that $S_R(\cdot)v_0$ still is continuous. So let $t \in [0,\infty)$ and $\varepsilon > 0$. There is $u_0 \in \mathcal{D}(R)$ with $\|v_0 - u_0\| < \varepsilon/3$. Since $S_R(\cdot)u_0$ is continuous there is $\delta > 0$ such that $\big\|S_R(s)u_0 - S_R(t)u_0\big\| < \varepsilon/3$ whenever $|s-t| < \delta$. Now from (3.11) it follows that

$$\begin{aligned} &\big\|S_R(s)v_0 - S_R(t)v_0\big\| \\ &\qquad \le\ \big\|S_R(s)(v_0-u_0)\big\| + \big\|S_R(s)u_0 - S_R(t)u_0\big\| + \big\|S_R(t)(u_0-v_0)\big\| \\ &\qquad \le\ 2\,\|v_0-u_0\| + \big\|S_R(s)u_0 - S_R(t)u_0\big\| \\ &\qquad <\ \varepsilon \end{aligned}$$

for $|s-t| < \delta$. □

Lemma 9. *For a maximal monotone operator $R : X \supseteq \mathcal{D}(R) \to X$ it holds*

$$\mathcal{D}(R) \quad = \quad \Big\{ v_0 \in X \ : \quad \lim_{h \to 0+} \frac{S_R(h)v_0 - S_R(0)v_0}{h} \quad \textit{exists.} \Big\} . \tag{3.12}$$

Proof. This is stated by Theorem 3.15 (Lumer, Phillips, 1961) in [10]. $\square$

Lemma 10. *For the semigroup $\big(S_R(t)\big)_{t\in[0,\infty)}$ defined by a maximal monotone operator $R : X \supseteq \mathcal{D}(R) \to X$ on a real Hilbert space $\big(X, (\,.\,,.)\big)$, the following holds true:*

(a) The map $[0, \infty) \times X \to X$, $(t, v_0) \mapsto S_R(t)v_0$ is continuous.

(b) $\frac{d}{dt} S_R(t)v_0 = -RS_R(t)v_0\,, \quad t \in [0, \infty),\ v_0 \in \mathcal{D}(R)\,.$

(c) $S_R(t)Rv_0 = RS_R(t)v_0\,, \quad t \in [0, \infty),\ v_0 \in \mathcal{D}(R)\,.$

(d) $\big\|RS_R(t)v_0\big\| \le \|Rv_0\|\,, \quad t \in [0, \infty),\ v_0 \in \mathcal{D}(R)\,.$

Proof. To prove part (a) we fix $t_0 \in [0, \infty)$ and $v_0 \in X$ and use the continuity of $S_R(\cdot)v_0$, which is implied by Lemma 8, as well as (3.11). This yields

$$\begin{aligned} \big\|S_R(t)v - S_R(t_0)v_0\big\| \ &\le\ \big\|S_R(t)(v - v_0)\big\| \ +\ \big\|S_R(t)v_0 - S_R(t_0)v_0\big\| \\ &\le\ \|v - v_0\| \ +\ \big\|S_R(t)v_0 - S_R(t_0)v_0\big\| \\ &\to\ 0\,, \qquad (t, v) \to (t_0, v_0)\,. \end{aligned}$$

Part (b) directly follows from the definition of $\big(S_R(t)\big)_{t\in[0,\infty)}$.
To prove (c), let $v_0 \in \mathcal{D}(R)$ and $t \in [0, \infty)$. Then

$$\begin{aligned} \lim_{h\to 0+} \frac{1}{h}\big(S_R(h)S_R(t)v_0 - S_R(t)v_0\big) \quad &= \quad \lim_{h\to 0+} \frac{1}{h}\big(S_R(h+t)v_0 - S_R(t)v_0\big) \\ = \quad \lim_{h\to 0+} \frac{1}{h}\big(S_R(t)S_R(h)v_0 - S_R(t)v_0\big) \quad &= \quad \lim_{h\to 0+} S_R(t)\Big(\frac{1}{h}\big(S_R(h)v_0 - v_0\big)\Big) \\ = \quad S_R(t)\Big(\lim_{h\to 0+} \frac{1}{h}\big(S_R(h)v_0 - v_0\big)\Big) \quad &= \quad -S_R(t)Rv_0 \end{aligned}$$

due to (b). So from (3.12) it follows that $S_R(t)v_0 \in \mathcal{D}(R)$ and again by (b) it is $-RS_R(0)S_R(t)v_0 = -S_R(t)Rv_0$, so $RS_R(t)v_0 = S_R(t)Rv_0$.

Finally, (d) is a consequence of (c) and (3.11). $\square$

3.2 The Inhomogeneous Equation

In this section we study the inhomogeneous evolution equation (3.1), that is

$$\begin{aligned} v'(t) &= -Rv(t) + f(t)\,, \qquad && t \in [0,t_1]\,, \\ v(0) &= v_0 && \end{aligned} \tag{3.13}$$

with a maximal monotone operator R on the real Hilbert space $\big(X, (\,.\,,.)\big)$ and v_0 and f having multiple degrees of regularity.

In the sequel we are going to need an integral notion for functions mapping a real interval into a Hilbert space and also some basic facts on integrable functions of this kind. To this end we refer to the appendix.

The most natural solution concept for (3.13) is the following.

Definition 11. *For $f \in C\big([0,t_1],\, X\big)$ and $v_0 \in \mathcal{D}(R)$ a function*

$$v \quad \in \quad C^1\big([0,t_1], X\big) \,\cap\, C\big([0,t_1], \mathcal{D}(R)\big)$$

which satisfies (3.13) *is called a* ***classical solution*** *of* (3.13). $\square$

To define a second notion of solutions of equation (3.13) we need the following basic technical lemma.

Lemma 12. *Let $\big(S_R(t)\big)_{t\in[0,\infty)}$ be as in Definition 7 and $f \in L^1\big((0,t_1), X\big)$. Then for every $t \in [0,t_1]$ it holds*

$$S_R(t-\bullet)f \quad \in \quad L^1\big((0,t), X\big)\,.$$

Proof. Let $t \in [0,t_1]$.

We need to prove that $g := S_R(t-\bullet)f$ is Bochner integrable as formulated in Definition A.1.

Since $f \in L^1\big((0,t_1), X\big)$ by assumption, there is a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions converging pointwise to f almost everywhere on $(0,t_1)$ and also it is $\int_0^{t_1} \|f(s)\|\, ds < \infty$. We define the functions

$$\tau_n \;:=\; \sum_{k=1}^{n-1} k\frac{t}{n}\, 1_{\left[k\frac{t}{n}\,,\,(k+1)\frac{t}{n}\right)}\,,$$

$n \in \mathbb{N}$, which have only finitely many values and map $(0,t)$ into itself. It is $\lim_{n\to\infty} \tau_n(s) = s,\ \ s \in (0,t)$. Now $g_n := S_R\big(t-\tau_n(\bullet)\big)f_n$ is a simple function for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} g_n(s) = g(s)$ for almost all $s \in (0,t)$ because of Lemma 10(a). So g is measurable. With (3.11) it holds

$$\|g(s) - g_n(s)\| \;\le\; \|g(s)\| + \|g_n(s)\| \;\le\; \|f(s)\| + \|f_n(s)\| \;\le\; 2\,\|f(s)\| + 1$$

for almost all $s \in (0,t)$ and n sufficiently large. Since $2\|f\|+1$ is integrable it follows from the theorem of dominated convergence that $\lim_{n\to\infty} \int_0^t \|g(s) - g_n(s)\| \, ds = 0$. Thus g is Bochner integrable. □

Definition 13. *For $f \in L^1\big((0,t_1), X\big)$ and $v_0 \in X$ the function*

$$v(t) \quad = \quad S_R(t)\, v_0 + \int_0^t S_R(t-s) f(s)\, ds\,, \qquad\qquad t \in [0,t_1]\,, \tag{3.14}$$

is called the ***mild solution*** *of* (3.13). □

By Lemma 12 the mild solution of (3.13) is well-defined.

Lemma 14. *The mild solution v defined in* (3.14) *has the properties*

$$v \quad \in \quad C\big([0,t_1],\, X\big)$$

and

$$\|v(t)\| \quad \le \quad \|v_0\| + \|f\|_{L^1((0,t),X)}\,, \qquad\qquad t \in [0,t_1]\,. \tag{3.15}$$

Proof. The continuity of the first summand in (3.14) follows from Lemma 10(a). To prove the continuity of the second summand we assume $t, s \in [0,t_1]$ and without loss of generality $s \le t$. Then

$$\begin{aligned}
&\Big\| \int_0^t S_R(t-\tau) f(\tau)\, d\tau - \int_0^s S_R(s-\tau) f(\tau)\, d\tau \Big\| \\
&\le \quad \Big\| \int_0^s \big(S_R(t-\tau) - S_R(s-\tau)\big) f(\tau)\, d\tau \Big\| \; + \; \Big\| \int_s^t S_R(t-\tau) f(\tau)\, d\tau \Big\| \\
&\le \quad \int_0^{t_1} \underbrace{\Big\| \big(S_R(t-\tau) - S_R(s-\tau)\big) f(\tau) \Big\|}_{\le \quad 2\|f(\tau)\|} \, d\tau \; + \; \int_s^t \|f(\tau)\|\, d\tau\,.
\end{aligned}$$

Now we apply the theorem of dominated convergence to see that for $s \to t$ and also for $t \to s$ both terms converge to 0.

Finally estimate (3.15) is proven by applying the triangle inequality to (3.14). □

The concept of mild solutions generalizes the notion of classical solutions as the following lemma shows.

Lemma 15. *Let $f \in C\big([0,t_1],\, X\big)$, $v_0 \in \mathcal{D}(R)$ and let v be a classical solution of* (3.13). *Then v also is a mild solution of* (3.13).

Proof. Let v be a classical solution of (3.13). For $t = 0$, (3.14) holds because of $S_R(0) = \mathrm{Id}$. For fixed $t \in (0, t_1]$, we define an auxiliary function

$$g(s) \quad := \quad S_R(t-s)v(s)\,, \qquad\qquad s \in [0,t]\,,$$

and prove that g is differentiable with

$$g'(s) \quad = \quad S_R(t-s)f(s)\,, \qquad\qquad s \in [0,t]\,. \tag{3.16}$$

Therefore we also fix $s \in [0,t]$. For $r \in [0,t] \setminus \{s\}$ we then have

$$\begin{aligned}
&\frac{1}{s-r}\,\big(g(s) - g(r)\big) \\
&\quad = \frac{1}{s-r}\,\big(S_R(t-s)v(s) - S_R(t-r)v(r)\big) \\
&\quad = \frac{1}{s-r}\,\Big(\big[S_R(t-s) - S_R(t-r)\big]v(s) \;+\; S_R(t-r)\big[v(s) - v(r)\big]\Big) \\
&\quad = \frac{1}{s-r}\,\big[S_R(t-s) - S_R(t-r)\big]v(s) \;+\; S_R(t-r)\Big(\frac{1}{s-r}\,\big[v(s) - v(r)\big]\Big) \\
&\quad \xrightarrow[r \to s]{} \quad S_R(t-s)Rv(s) \;+\; S_R(t-s)v'(s) \\
&\qquad\qquad\qquad = S_R(t-s)Rv(s) \;+\; S_R(t-s)\big[-Rv(s) + f(s)\big] \\
&\qquad\qquad\qquad = S_R(t-s)f(s)\,.
\end{aligned}$$

Here the first summand converges according to Lemma 10(b) and the second summand converges because of Lemma 10(a).

Now we integrate (3.16) to get

$$v(t) \;-\; S_R(t)v_0 \quad = \quad g(t) \;-\; g(0) \quad = \quad \int_0^t g'(s)\,ds \quad = \quad \int_0^t S_R(t-s)f(s)\,ds$$

which concludes the proof. □

Remark 16. *By construction, a mild solution of* (3.13) *trivially is unique.* □

Next we prove the existence of a classical solution of (3.13) provided the right-hand side f is an element of $W^{1,1}\big((0,t_1),\,X\big)$.

Lemma 17. *Let* $f \in W^{1,1}\big((0,t_1),\,X\big)$, $v_0 \in \mathcal{D}(R)$ *and let* v *be the mild solution of* (3.13)*. Then* v *also is a classical solution of* (3.13)*.*

Furthermore, with the graph norm

$$\|x\|_R \quad = \quad \|x\| \;+\; \|Rx\|\,, \qquad\qquad x \in \mathcal{D}(R)\,,$$

of the operator R it holds

$$\frac{d}{dt}v(t) \quad = \quad S_R(t)\big[f(0) - Rv_0\big] + \int_0^t S_R(t-s)f'(s)\,ds\,, \qquad t \in [0,t_1]\,, \tag{3.17}$$

$$\Big\|\frac{dv}{dt}(t)\Big\| \quad \leq \quad \|f(0) - Rv_0\| + \big\|f'\big\|_{L^1((0,t),X)}\,, \qquad t \in [0,t_1]\,, \tag{3.18}$$

$$\|v(t)\|_R \quad \leq \quad \|v_0\|_R \;+\; (2\,c_{CW} + 1)\;\|f\|_{W^{1,1}((0,t_1),X)}\,, \qquad t \in [0,t_1]\,, \tag{3.19}$$

with $c_{CW} = \max\{1/t_1, 1\}$ *from* (A.1).

Proof. First we show that $v \in C^1\big([0,t_1],X\big) \cap C\big([0,t_1],\mathcal{D}(R)\big)$. As $v_0 \in \mathcal{D}(R)$ the first summand $S_R(\bullet)v_0$ of (3.14) is an element of $C^1\big([0,t_1],X\big) \cap C\big([0,t_1],\mathcal{D}(R)\big)$ by definition of S_R, and with Lemma 10 (b) and (c) it holds

$$\frac{d}{dt}S_R(t)v_0 \quad = \quad -S_R(t)Rv_0\,, \qquad\qquad t \in [0,t_1]\,. \tag{3.20}$$

By applying the variable substitution $s' = t-s$, the second summand of (3.14) can be written as $\int_0^t S_R(s)f(t-s)\,ds$. To show that $\int_0^\bullet S_R(s)f(\bullet - s)\,ds$ is an element of $C^1\big([0,t_1],X\big)$ and that

$$\frac{d}{dt}\int_0^t S_R(s)f(t-s)\,ds \quad = \quad S_R(t)f(0) \;+\; \int_0^t S_R(s)f'(t-s)\,ds\,, \tag{3.21}$$

$t \in [0,t_1]$, holds, we approximate f in $W^{1,1}\big((0,t_1),X\big)$ by smooth functions $(\varphi_n)_{n\in\mathbb{N}}$ according to Lemma A.4. Then (3.21) holds for f being substituted by φ_n for every $n \in \mathbb{N}$. It is

$$\begin{aligned}\Big\|\int_0^t S_R(s)\varphi_n(t-s)\,ds \;-\; \int_0^t S_R(s)\varphi_m(t-s)\,ds\Big\| \\ &\leq \int_0^t \big\|\varphi_n(t-s) - \varphi_m(t-s)\big\|\,ds \\ &\leq \|\varphi_n - \varphi_m\|_{L^1((0,t_1),X)}\,,\end{aligned}$$

$t \in [0,t_1]$, and with Lemma A.2 it also holds

$$\begin{aligned}&\Big\|\frac{d}{dt}\int_0^t S_R(s)\varphi_n(t-s)\,ds \;-\; \frac{d}{dt}\int_0^t S_R(s)\varphi_m(t-s)\,ds\Big\| \\ &\quad\leq \big\|S_R(t)\big(\varphi_n(0) - \varphi_m(0)\big)\big\| \;+\; \int_0^t \big\|S_R(s)\big(\varphi_n'(t-s) - \varphi_m'(t-s)\big)\big\|\,ds \\ &\quad\leq \big\|\varphi_n(0) - \varphi_m(0)\big\| \;+\; \int_0^t \big\|\varphi_n'(t-s) - \varphi_m'(t-s)\big\|\,ds \\ &\quad\leq (c_{CW} + 1)\,\|\varphi_n - \varphi_m\|_{W^{1,1}((0,t_1),X)}\,,\end{aligned}$$

$t \in [0,t_1]$. So $\big(\int_0^\bullet S_R(s)\varphi_n(\bullet - s)\,ds\big)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C^1\big([0,t_1],X\big)$ and therefore convergent with

$$\begin{aligned}\Big(\int_0^\bullet S_R(s)f(\bullet - s)\,ds\Big)' &= \Big(\lim_{n\to\infty}\int_0^\bullet S_R(s)\varphi_n(\bullet - s)\,ds\Big)' \\ &= \lim_{n\to\infty}\Big(\int_0^\bullet S_R(s)\varphi_n(\bullet - s)\,ds\Big)' \\ &= \lim_{n\to\infty}\Big(S_R(\bullet)\varphi_n(0) \;+\; \int_0^\bullet S_R(s)\varphi_n'(\bullet - s)\,ds\Big) \\ &= S_R(\bullet)f(0) \;+\; \int_0^\bullet S_R(s)f'(\bullet - s)\,ds\,,\end{aligned}$$

where the limits are taken with respect to the norm $\|.\|_{C([0,t_1],X)}$. Hence we have $v \in C^1\big([0,t_1],X\big)$ and (3.21) which in combination with the variable substitution $s' = t - s$ in the integrand and (3.20) yields (3.17).

Next we prove that for every $t \in [0,t_1]$ the second summand of (3.14) is an element of $\mathcal{D}(R)$. Because of (3.12) we calculate

$$\begin{aligned}&\frac{1}{h}\big[S_R(h) - \mathrm{Id}\big]\int_0^t S_R(t-s)f(s)\,ds \\ &\qquad = \frac{1}{h}\Big(\int_0^{t+h} S_R(t+h-s)f(s)\,ds - \int_0^t S_R(t-s)f(s)\,ds\Big) \\ &\qquad\qquad - \frac{1}{h}\int_t^{t+h} S_R(t+h-s)f(s)\,ds\,,\end{aligned}$$

$t \in [0,t_1]$, $h > 0$. For $h \to 0$ the first summand converges to $S_R(t)f(0) + \int_0^t S_R(s)f'(t-s)\,ds$ due to (3.21).

The second summand converges as well which can be seen as follows. Let $\varepsilon > 0$. Because f and $(s,x) \mapsto S_R(s)x$ are continuous there is $\delta > 0$ such that for $t \le s \le t+h < t+\delta$ it holds

$$\|S_R(t+h-s)f(s) - f(t)\| \;=\; \|S_R(t+h-s)f(s) - S_R(0)f(t)\| \;<\; \varepsilon\,.$$

For those $h < \delta$ it then also holds

$$\begin{aligned}&\Big\|\frac{1}{h}\int_t^{t+h} S_R(t+h-s)f(s)\;ds \;-\; f(t)\Big\| \\ &\qquad = \Big\|\frac{1}{h}\int_t^{t+h} S_R(t+h-s)f(s) - f(t)\;ds\Big\| \\ &\qquad \le \frac{1}{h}\int_t^{t+h} \big\|S_R(t+h-s)f(s) - f(t)\big\|\;ds\end{aligned}$$

$$
\begin{aligned}
&\leq \quad \frac{1}{h}\int_t^{t+h} \varepsilon \, ds \\
&= \quad \varepsilon \, .
\end{aligned}
$$

So also the second summand converges and its limit is $-f(t)$. Thus $v(t) \in \mathcal{D}(R)$, and Lemma 10(b), where $t = 0$, together with Lemma 10(c) yield

$$
\begin{aligned}
-Rv(t) &= -S_R(t)Rv_0 + S_R(t)f(0) + \int_0^t S_R(s)f'(t-s)\,ds - f(t) \\
&= S_R(t)\big(f(0) - Rv_0\big) + \int_0^t S_R(s)f'(t-s)\,ds - f(t) \\
&= v'(t) - f(t)\,,
\end{aligned}
$$

$t \in [0, t_1]$, where the last equality follows from (3.17). From this equality and $v(0) = v_0$ we see that v solves (3.13) and also that $v \in C\big([0, t_1], (\mathcal{D}(R), \|.\|_R)\big)$. Hence v is a classical solution of (3.13).

Now estimate (3.18) is proven by using the triangle inequality together with $\|S_R(s)\|_{\mathcal{L}(X)} \leq 1,\ s \in [0, \infty)$. Estimate (3.19) is proven like this:

$$
\begin{aligned}
\|v(t)\|_R &= \|Rv(t)\| + \|v(t)\| \quad \leq \quad \Big\|\frac{dv}{dt}(t)\Big\| + \|f(t)\| + \|v(t)\| \\
&\leq \|f(0) - Rv_0\| + \big\|f'\big\|_{L^1((0,t),X)} \\
&\quad + \|f(t)\| + \|v_0\| + \|f\|_{L^1((0,t),X)} \\
&\leq \|v_0\|_R + (2\,c_{CW} + 1)\,\|f\|_{W^{1,1}((0,t_1),X)}\,,
\end{aligned}
$$

$t \in [0, t_1]$, where we used (3.15), (3.18) and (A.1). □

Corollary 18. *Let $f \in W^{1,1}\big((0, t_1),\, X\big)$ and $v_0 \in \mathcal{D}(R)$. Then the inhomogeneous evolution equation* (3.13) *has a unique classical solution.*

Proof. The statement of this corollary is a combination of the first part of Lemma 17 and Remark 16. □

Example 19. If we only have $f \in C\big([0, t_1],\, X\big)$ and $v_0 \in \mathcal{D}(R)$, a classical solution of (3.13) does not have to exist as this example will show. It can be found in [10], Example 7.9.

Here we assume, that $\big(S_R(t)\big)_{t\in\mathbb{R}}$ even is a group, $z_0 \in X \setminus \mathcal{D}(R)$ and $f(t) := S_R(t)z_0,\ t \in \mathbb{R}$. Then f is continuous but not differentiable in any point $t_0 \in \mathbb{R}$. Otherwise $S_R(\cdot)z_0$ would also be differentiable in $t = 0$, which in the case $t_0 > 0$ can be seen by time reversal since $\big(S_R(t)\big)_{t\in\mathbb{R}}$ is a group. And this would be a contradiction to $z_0 \notin \mathcal{D}(R)$ and (3.12). Now

$$\begin{aligned} v(t) &:= \int_0^t S_R(t-s)f(s)\,ds = \int_0^t S_R(t-s)S_R(s)z_0\,ds \\ &= \int_0^t S_R(t)z_0\,ds = \int_0^t f(t)\,ds = tf(t)\,, \qquad t \in [0,t_1]\,, \end{aligned}$$

is the mild solution of

$$v'(t) = -Rv(t) + f(t)\,, \qquad t \in [0,t_1]\,, \qquad v(0) = 0\,.$$

And v is not differentiable at any point $t > 0$. Otherwise also $f(t) = \frac{1}{t}v(t)$ would be differentiable as a function of t. But we have seen in the beginning of this example, that it is not. □

Lemma 20. *Let $f \in W^{2,1}\big((0,t_1),X\big)$, $v_0 \in \mathcal{D}(R)$ and $f(0) - Rv_0 \in \mathcal{D}(R)$. Then for the mild solution v of (3.13), which by Lemma 17 is also classical, it even holds $v \in C^2\big([0,t_1],X\big) \cap C^1\big([0,t_1],\mathcal{D}(R)\big)$. Furthermore, we have*

$$\begin{aligned} \|v\|_{C^1([0,t_1],\mathcal{D}(R))} \leq\ & \|v_0\|_R + \|f(0) - Rv_0\|_R \\ & + (4\,c_{CW} + 2)\,\|f\|_{W^{2,1}((0,t_1),X)} \end{aligned} \tag{3.22}$$

with $c_{CW} = \max\{1/t_1, 1\}$ from (A.1).

Proof. With $\tilde{f} := f'$, $\tilde{v}_0 := f(0) - Rv_0 \in \mathcal{D}(R)$ and $\tilde{v} := v'$ equation (3.17) can be written in the form $\tilde{v}(t) = S_R(t)\tilde{v}_0 + \int_0^t S_R(t-s)\tilde{f}(s)\,ds$, $t \in [0,t_1]$. That means $\tilde{v}$ is the mild solution of

$$\tilde{v}' + R\tilde{v} = \tilde{f}\,, \qquad \tilde{v}(0) = \tilde{v}_0\,. \tag{3.23}$$

With $f \in W^{2,1}\big((0,t_1),X\big)$ it holds $\tilde{f} \in W^{1,1}\big((0,t_1),X\big)$. So from Lemma 17 it follows that

$$\tilde{v} \in C^1\big([0,t_1],X\big) \cap C\big([0,t_1],\mathcal{D}(R)\big) \tag{3.24}$$

and $\tilde{v}$ is even a classical solution of (3.23).

Because $\tilde{v}$ is the derivative of v with respect to the norm $\|.\|$ it follows immediately from (3.24) that $v \in C^2\big([0,t_1],X\big)$. We need to prove that v is also differentiable with respect to the norm $\|.\|_R$. Then it also follows from (3.24) that $v \in C^1\big([0,t_1],\mathcal{D}(R)\big)$. So we calculate

$$\begin{aligned} &\left\|\frac{v(t+h) - v(t)}{h} - \tilde{v}(t)\right\|_R \\ &\qquad = \left\|\frac{v(t+h) - v(t)}{h} - \tilde{v}(t)\right\| + \left\|\frac{Rv(t+h) - Rv(t)}{h} - R\tilde{v}(t)\right\|, \end{aligned}$$

$t,\, t+h \in [0,t_1]$. For $h \to 0$ the first summand converges to 0 by the definition of $\tilde{v}$. Using $v' = -Rv + f$, the second summand can be written

$$\begin{aligned} &\Big\| \frac{Rv(t+h) - Rv(t)}{h} - R\tilde{v}(t) \Big\| \\ &\qquad = \Big\| \frac{-v'(t+h) + v'(t)}{h} + \frac{f(t+h) - f(t)}{h} - R\tilde{v}(t) \Big\| , \end{aligned} \tag{3.25}$$

$t,\, t+h \in [0,t_1]$. As $f' \in W^{1,1}\big((0,t_1),X\big) \hookrightarrow C\big([0,t_1],X\big)$ by Lemma A.2, it is $f \in C^1\big([0,t_1],X\big)$. Taking furthermore into account $v \in C^2\big([0,t_1],X\big)$ we see that for $h \to 0$ the right-hand side of (3.25) tends to

$$\big\| - v''(t) + f'(t) - R\tilde{v}(t) \big\| \quad = \quad \big\| - \tilde{v}'(t) + \tilde{f}(t) - R\tilde{v}(t) \big\| \quad = \quad 0 \, ,$$

$t \in [0,t_1]$, where the last equality follows from (3.23).

Finally, (3.19) yields

$$\begin{aligned} \|v'(t)\|_R \quad &= \quad \|\tilde{v}(t)\|_R \\ &\leq \quad \|\tilde{v}_0\|_R \;+\; (2\,c_{CW} + 1)\; \|\tilde{f}\|_{W^{1,1}((0,t_1),X)} \\ &\leq \quad \|f(0) - Rv_0\|_R \;+\; (2\,c_{CW} + 1)\; \|f'\|_{W^{1,1}((0,t_1),X)} \, , \end{aligned}$$

$t \in [0,t_1]$. Again with (3.19) applied to v, estimate (3.22) follows. □

A more general statement than Lemma 20 and its proof can be found in [14], Theorem 2.6.

Chapter 4

Abstract Auxiliary Results

4.1 An Abstract Variable Transformation

Throughout this section let $\big(X, (.\,,.)_X\big)$ denote a real Hilbert space, $\|.\|_X$ the norm induced by $(.\,,.)_X$ and let $t_1 > 0$.

In section 3.1 we saw, that if $R : X \supseteq \mathcal{D}(R) \to X$ is maximal monotone with respect to a scalar product $(.\,,.)$ on X with corresponding norm $\|.\|$, the evolution equation

$$\begin{aligned} v'(t) &= -Rv(t)\,, \qquad && t \in [0, t_1]\,, \\ v(0) &= v_0\,, \end{aligned}$$

has a unique solution in the space $C^1\Big([0, t_1], \big(X, \|.\|\big)\Big) \cap C\Big([0, t_1], \big(\mathcal{D}(R), \|.\|_R\big)\Big)$. To find a scalar product $(.\,,.)$ with this property for some given R, it can be helpful to subject the unknown v to a variable transformation. What this means on an abstract level, is what we would like to study in this section.

Definition 21. *Let $R_1 : X \supseteq \mathcal{D}(R_1) \to X$, $R_2 : X \supseteq \mathcal{D}(R_2) \to X$ be not necessarily bounded, linear operators with domain of definition $\mathcal{D}(R_1)$ and $\mathcal{D}(R_2)$ respectively and $S : X \to X$ a bounded and boundedly invertible linear operator. Furthermore, let $(.\,,.)_1$ and $(.\,,.)_2$ be two scalar products on X. We call the pairs $\big(R_1\,, (.\,,.)_1\big)$ and $\big(R_2\,, (.\,,.)_2\big)$* ***similar*** *(via S), iff*

$$\begin{aligned} R_2 &= SR_1S^{-1} \\ \textit{and} \qquad \mathcal{D}(R_2) &= S\big(\mathcal{D}(R_1)\big) \\ \textit{and} \qquad (u_1, u_2)_1 &= (Su_1,\, Su_2)_2\,, \qquad u_1, u_2 \in X. \end{aligned} \tag{4.1}$$

□

Lemma 22. *Similarity as defined in Definition 21 is an equivalence relation.*

Proof. To check the reflexivity of this relation we take $S = \mathrm{Id}$ in the notation of Definition 21. Symmetry holds because if $\big(R_1\,,\,(.\,,.)_1\big)$ and $\big(R_2\,,\,(.\,,.)_2\big)$ are similar via S, then we easily calculate that $\big(R_2\,,\,(.\,,.)_2\big)$ and $\big(R_1\,,\,(.\,,.)_1\big)$ are similar via S^{-1}. And finally, similarity is transitive because if $\big(R_1\,,\,(.\,,.)_1\big)$ and $\big(R_2\,,\,(.\,,.)_2\big)$ are similar via S_1 and $\big(R_2\,,\,(.\,,.)_2\big)$ and $\big(R_3\,,\,(.\,,.)_3\big)$ are similar via S_2 it obviously holds that $\big(R_1\,,\,(.\,,.)_1\big)$ and $\big(R_3\,,\,(.\,,.)_3\big)$ are similar via S_2S_1. $\square$

Throughout this section we assume the following.

Assumption 23. Let $A : X \supseteq \mathcal{D}(A) \to X$ be a linear not necessarily bounded operator with domain of definition $\mathcal{D}(A)$, $T : X \to X$ an invertible bounded linear operator (then by the open mapping theorem also $T^{-1} \in \mathcal{L}(X)$) and $(.\,,.)_E$ an alternative scalar product on X which is equivalent to $(.\,,.)_X$. In addition we define the linear operator $B := TAT^{-1}$ with domain of definition $\mathcal{D}(B) := T\big(\mathcal{D}(A)\big)$ and the scalar product

$$(u,v)_T \quad := \quad (Tu,\, Tv)_E\,, \qquad\qquad u,\, v \in X\,. \tag{4.2}$$

So $\big(A\,,\,(.\,,.)_T\big)$ and $\big(B\,,\,(.\,,.)_E\big)$ are similar via T. $\square$

Lemma 24. $(.\,,.)_T$ *defined in* (4.2) *actually is a scalar product on* X *which is equivalent to* $(.\,,.)_X$. *With* $\|.\|_E$ *and* $\|.\|_T$ *denoting the norms induced by* $(.\,,.)_E$ *and* $(.\,,.)_T$*, respectively, also the pairs* $\|\cdot\|_{B,X} := \|\cdot\|_X + \|B\cdot\|_X$ *and* $\|\cdot\|_{B,E} := \|\cdot\|_E + \|B\cdot\|_E$ *as well as* $\|\cdot\|_{A,X} := \|\cdot\|_X + \|A\cdot\|_X$ *and* $\|\cdot\|_{A,T} := \|\cdot\|_T + \|A\cdot\|_T$ *of graph norms are equivalent.*

Furthermore,

$$\begin{aligned} &C^1\Big([0,t_1],(X,\|.\|_E)\Big) \cap C\Big([0,t_1],\big(\mathcal{D}(B),\|.\|_{B,E}\big)\Big) \\ &\quad = \quad C^1\Big([0,t_1],(X,\|.\|_X)\Big) \cap C\Big([0,t_1],\big(\mathcal{D}(B),\|.\|_{B,X}\big)\Big) \end{aligned}$$

and

$$\begin{aligned} &C^1\Big([0,t_1],(X,\|.\|_T)\Big) \cap C\Big([0,t_1],\big(\mathcal{D}(A),\|.\|_{A,T}\big)\Big) \\ &\quad = \quad C^1\Big([0,t_1],(X,\|.\|_X)\Big) \cap C\Big([0,t_1],\big(\mathcal{D}(A),\|.\|_{A,X}\big)\Big)\,. \end{aligned}$$

Therefore we simply write $C^1\big([0,t_1],X\big) \cap C\big([0,t_1],\mathcal{D}(B)\big)$*, etc. in this situation, since the specific norm does not matter.*

Proof. As $(\cdot\,,\cdot)_E$ is a scalar product also $(\cdot\,,\cdot)_T$ is symmetric and positive semi-definite, and its bilinearity follows from the linearity of T. Now let $u \in X$ with $(u,u)_T = 0$. Then $Tu = 0$ and as by definition T is one-to-one, it follows that $u = 0$. So $(\cdot\,,\cdot)_T$ is definite and hence a scalar product.

Since $(.\,,.)_E$ is equivalent to $(.\,,.)_X$ there are $c_E, C_E > 0$ with

$$c_E\,\|w\|_X \quad\leq\quad \|w\|_E \quad\leq\quad C_E\,\|w\|_X\,, \qquad\qquad w \in X\,.$$

Together with the boundedness of T and T^{-1} with respect to $\|\cdot\|_X$ it follows

$$\|u\|_T \quad=\quad \|Tu\|_E \quad\leq\quad C_E\,\|Tu\|_X \quad\leq\quad C_E\,\|T\|_{\mathcal{L}(X,\|.\|_X)}\,\|u\|_X$$

and

$$\begin{aligned} c_E\,\|T^{-1}\|^{-1}_{\mathcal{L}(X,\|.\|_X)}\,\|u\|_X \quad&=\quad c_E\,\|T^{-1}\|^{-1}_{\mathcal{L}(X,\|.\|_X)}\,\|T^{-1}Tu\|_X \quad\leq\quad c_E\,\|Tu\|_X \\ &\leq\quad \|Tu\|_E \quad=\quad \|u\|_T\,, \end{aligned}$$

$u \in X$. So also $(.\,,.)_T$ is equivalent to $(.\,,.)_X$.

All other statements are direct consequences. □

Lemma 25. *The operator*

$$T : \big(X, \|.\|_T\big) \to \big(X, \|.\|_E\big)$$

as well as its restriction

$$T : \big(\mathcal{D}(A), \|.\|_{A,T}\big) \to \big(\mathcal{D}(B), \|.\|_{B,E}\big)$$

are isometries.

Proof. Since $T : X \to X$ is bijective the first statement holds due to the definition of $\|.\|_T$. The second statement follows from the injectivity of T, the definition of $\mathcal{D}(B)$ and the definition of B since

$$\begin{aligned} \|Tu\|_{B,E} \;&=\; \|Tu\|_E \,+\, \|BTu\|_E \;=\; \|Tu\|_E \,+\, \|TAu\|_E \;=\; \|u\|_T \,+\, \|Au\|_T \\ &=\; \|u\|_{A,T}\,, \end{aligned}$$

$u \in \mathcal{D}(A)$. □

Lemma 26. *The linear map*

$$\begin{aligned} \tilde{T}\;:\quad & C^1\Big([0,t_1], (X,\|.\|_T)\Big) \cap C\Big([0,t_1], \big(\mathcal{D}(A), \|.\|_{A,T}\big)\Big) \\ &\qquad\to\quad C^1\Big([0,t_1], (X,\|.\|_E)\Big) \cap C\Big([0,t_1], \big(\mathcal{D}(B), \|.\|_{B,E}\big)\Big)\,, \\ \big(\tilde{T}u\big)(t) \quad&:=\quad T\big(u(t)\big)\,, \qquad\qquad t \in [0,t_1]\,, \end{aligned} \tag{4.3}$$

is well-defined and isometric with respect to the pairs of norms

$$u \mapsto \max_{t\in[0,t_1]}\Big(\big\|u(t)\big\|_T + \big\|u'(t)\big\|_T\Big) \qquad \textit{and} \qquad w \mapsto \max_{t\in[0,t_1]}\Big(\big\|w(t)\big\|_E + \big\|w'(t)\big\|_E\Big)$$

as well as

$$u \mapsto \max_{t\in[0,t_1]} \big\|u(t)\big\|_{A,T} \qquad \textit{and} \qquad w \mapsto \max_{t\in[0,t_1]} \big\|w(t)\big\|_{B,E} .$$

The derivative of $\tilde{T}u$ with respect to t is given by

$$\big(\tilde{T}u\big)'(t) \quad = \quad T\big(u'(t)\big)\,, \qquad\qquad t \in [0,t_1]\,. \tag{4.4}$$

Proof. Since the statement of this lemma even holds true for unbounded transformations T, we give a proof for this more general case.

Let $u \in C^1\Big([0,t_1],(X,\|.\|_T)\Big) \cap C\Big([0,t_1],\big(\mathcal{D}(A),\|.\|_{A,T}\big)\Big)$. We need to prove that $\tilde{T}u \in C^1\Big([0,t_1],(X,\|.\|_E)\Big) \cap C\Big([0,t_1],\big(\mathcal{D}(B),\|.\|_{B,E}\big)\Big)$.

First we prove the differentiability of $\tilde{T}u$. Let $t_0 \in [0,t_1]$. From the differentiability of u with respect to $\|.\|_T$ and Lemma 25 it follows

$$\begin{aligned}
\lim_{\substack{t\to t_0\\ \|\cdot\|_E}} \frac{\big(\tilde{T}u\big)(t) - \big(\tilde{T}u\big)(t_0)}{t-t_0} \quad &= \quad \lim_{\substack{t\to t_0\\ \|\cdot\|_E}} \frac{T\big(u(t)\big) - T\big(u(t_0)\big)}{t-t_0} \\
&= \quad \lim_{\substack{t\to t_0\\ \|\cdot\|_E}} T\left(\frac{u(t)-u(t_0)}{t-t_0}\right) \\
&= \quad T \lim_{\substack{t\to t_0\\ \|\cdot\|_T}} \frac{u(t)-u(t_0)}{t-t_0} \\
&= \quad T\left(u'(t_0)\right),
\end{aligned}$$

which in particular proves (4.4).

Furthermore this derivative is continuous: For $t_0 \in [0,t_1]$ we have

$$\lim_{t\to t_0} \big\|T\left(u'(t)\right) - T\left(u'(t_0)\right)\big\|_E \quad = \quad \lim_{t\to t_0} \big\|T\left(u'(t) - u'(t_0)\right)\big\|_E \quad = \quad 0$$

by the continuity of u' with respect to $\|.\|_T$ and Lemma 25 again. So $\tilde{T}u \in C^1\big([0,t_1],(X,\|.\|_E)\big)$.

Secondly we prove $\tilde{T}u \in C\Big([0,t_1],\big(\mathcal{D}(B),\|.\|_{B,E}\big)\Big)$. Let $t_0 \in [0,t_1]$. Then

$$\lim_{t\to t_0} \big\|\big(\tilde{T}u\big)(t) - \big(\tilde{T}u\big)(t_0)\big\|_{B,E} \quad = \quad \lim_{t\to t_0} \big\|T\big(u(t)-u(t_0)\big)\big\|_{B,E} \quad = \quad 0\,,$$

which follows from $u \in C\Big([0,t_1],\big(\mathcal{D}(A),\|.\|_{A,T}\big)\Big)$ and the second statement of Lemma 25.

Now by finally using Lemma 25 again we find that $\tilde{T}$ is isometric with respect to the two pairs of norms in question. $\square$

Theorem 27. *Let $u_0 \in \mathcal{D}(A)$ and $f : [0, t_1] \to X$. A function $u \in C^1\big([0,t_1], X\big) \cap C\big([0,t_1], \mathcal{D}(A)\big)$ solves*

$$\begin{aligned} u'(t) &= -A\big(u(t)\big) + f(t)\,, \qquad & t \in [0,t_1]\,, \\ u(0) &= u_0\,, \end{aligned} \tag{4.5}$$

iff $w := \tilde{T}u \in C^1\big([0,t_1], X\big) \cap C\big([0,t_1], \mathcal{D}(B)\big)$ solves the transformed evolution equation

$$\begin{aligned} w'(t) &= -B\big(w(t)\big) + T\big(f(t)\big)\,, \qquad & t \in [0,t_1]\,, \\ w(0) &= Tu_0\,. \end{aligned} \tag{4.6}$$

Proof. Let $u_0 \in \mathcal{D}(A)$ and $u \in C^1\big([0,t_1], X\big) \cap C\big([0,t_1], \mathcal{D}(A)\big)$. On the one hand, by (4.4), we have

$$\big(\tilde{T}u\big)'(t) \quad = \quad T\left(u'(t)\right), \qquad t \in [0,t_1]\,, \tag{4.7}$$

and on the other hand, by the definition of B and $\tilde{T}$, we get

$$T\Big(-A\big(u(t)\big)\Big) = -TA\left(u(t)\right) = -BT\left(u(t)\right) = -B\left(\big(\tilde{T}u\big)(t)\right), \tag{4.8}$$

$t \in [0,t_1]$. Furthermore, by the definition of $\tilde{T}$, it holds

$$\big(\tilde{T}u\big)(0) \quad = \quad T\big(u(0)\big)\,. \tag{4.9}$$

So if u solves (4.5), then $u' = -Au + f$ and $u(0) = u_0$. Hence applying T for every fixed $t \in [0,t_1]$ to both sides of (4.5) together with (4.7) – (4.9) yields that $\tilde{T}u$ solves (4.6).

If vice versa $\tilde{T}u$ solves (4.6), then $(\tilde{T}u)'(t) = -B(\tilde{T}u)(t) + T(f(t))$, $t \in [0,t_1]$, and $(\tilde{T}u)(0) = Tu_0$. Thus by applying T^{-1} for every fixed $t \in [0,t_1]$ to (4.6) together with (4.7) – (4.9), we find that u solves (4.5). □

Theorem 28. *In the sense of Definition 1, the operator $A : X \supseteq \mathcal{D}(A) \to X$ is maximal monotone with respect to $(.\,,.)_T$, iff $B : X \supseteq \mathcal{D}(B) \to X$ is maximal monotone with respect to $(.\,,.)_E$.*

Proof. Let $B : X \supseteq \mathcal{D}(B) \to X$ be maximal monotone with respect to $(.\,,.)_E$. Then

$$(Au, u)_T = \big(TAu,\, Tu\big)_E = \big(BTu,\, Tu\big)_E \geq 0\,, \qquad u \in \mathcal{D}(A)\,.$$

So $A : X \supseteq \mathcal{D}(A) \to X$ is monotone with respect to $(.\,,.)_T$.

Furthermore for each $f \in X$ there is $w \in \mathcal{D}(B)$ with $w + Bw = Tf$. So $TT^{-1}w + BTT^{-1}w = Tf$ and therefore $TT^{-1}w + TAT^{-1}w = Tf$. Application of T^{-1} yields $T^{-1}w + AT^{-1}w = f$, where $T^{-1}w \in \mathcal{D}(A)$. So $\mathrm{Id} + A : \mathcal{D}(A) \to X$ is onto. Overall $A : X \supseteq \mathcal{D}(A) \to X$ is maximal monotone with respect to $(.\,,.)_T$.

The opposite direction follows from the symmetry of the similarity relation which by Lemma 22 is an equivalence relation. □

Remark 29. *In our application the operator T will describe a transformation of variables which transforms a concrete partial differential equation represented by* (4.5) *into another one represented by* (4.6). *Both describe the same physical process using different variables.*

The physical energy of the state of the process corresponding to the solution u of (4.5) *at one point in time t is given by the expression $\frac{1}{2}\|u(t)\|_T^2$. Described in the other variables this physical process corresponds to the solution $w = \tilde{T}u$ of* (4.6). *Here, the physical energy at time t has the form $\frac{1}{2}\|w(t)\|_E^2$. Now Lemma 25 guarantees that*

$$\frac{1}{2}\|u(t)\|_T^2 \;=\; \frac{1}{2}\|w(t)\|_E^2\,.$$

So the physical energy of this state is well defined no matter which system of variables we use. □

Also we will have to apply a variable transformation to the derivative of the solution of an evolution equation for parameters contained in the equation. As we will see such a derivative is represented by the mild solution of an evolution equation. For this purpose we need the following two corollaries.

Corollary 30. *If A is maximal monotone with respect to $(.\,,.)_T$ in the sense of Definition 1 and $S_A(\cdot)$ the contraction semigroup generated by it according to Definition 7, then*

$$S_{TAT^{-1}}(t) \quad = \quad TS_A(t)T^{-1}\,, \qquad\qquad t \in [0,\infty)\,.$$

Proof. By Theorem 28 the operator $TAT^{-1} = B$ is maximal monotone with respect to $(.\,,.)_E$.

From Theorem 27 it follows that

$$\big(TS_A(t)T^{-1}\big)(Tu_0) \;=\; TS_A(t)u_0 \;=\; S_{TAT^{-1}}(t)(Tu_0)\,, \qquad t \in [0,\infty)\,, \tag{4.10}$$

$u_0 \in \mathcal{D}(A)$. As a consequence of the maximal monotonicity of B with respect to $(.\,,.)_E$, it is $T\mathcal{D}(A) = \mathcal{D}(B)$ dense in X. Together with the continuity of $TS_A(t)T^{-1}$ and $S_{TAT^{-1}}(t)$ it therefore follows from (4.10), that

$$\big(TS_A(t)T^{-1}\big)w_0 \;=\; S_{TAT^{-1}}(t)w_0\,, \qquad t \in [0,\infty)\,,$$

$w_0 \in X$. □

Corollary 31. *Let A be maximal monotone with respect to $(.\,,.)_T$, $u_0 \in X$ and $f \in L^1\big((0,t_1),X\big)$. A function $u \in C\big([0,t_1],X\big)$ is the mild solution of the evolution equation*

$$\begin{aligned} u'(t) &= -A\big(u(t)\big) + f(t)\,, \qquad && t \in [0,t_1]\,, \\ u(0) &= u_0\,, \end{aligned} \tag{4.11}$$

iff $w := \tilde{T}u \in C\big([0,t_1],X\big)$ is the mild solution of the transformed evolution equation

$$\begin{aligned} w'(t) &= -B\big(w(t)\big) + T\big(f(t)\big)\,, \qquad && t \in [0,t_1]\,, \\ w(0) &= Tu_0\,. \end{aligned} \tag{4.12}$$

Proof. Let u be the mild solution of (4.11). Then $u(t) = S_A(t)u_0 + \int_0^t S_A(t-s)f(s)\,ds$, $t \in [0,t_1]$. And with the continuity of T and Corollary 30 we get

$$\begin{aligned} (\tilde{T}u)(t) &= T\big(u(t)\big) \\ &= TS_A(t)T^{-1}Tu_0 + \int_0^t TS_A(t-s)T^{-1}T\big(f(s)\big)\,ds \\ &= S_B(t)Tu_0 + \int_0^t S_B(t-s)T\big(f(s)\big)\,ds\,, \end{aligned}$$

$t \in [0,t_1]$. So $\tilde{T}u$ is the mild solution of (4.12).

For the opposite direction we apply $\tilde{T}^{-1}$ to the mild solution of (4.12) and use Corollary 30 with A, B and T, T^{-1} interchanged, respectively. □

Finally, we will derive adjoints of given bounded and unbounded linear operators in our application. In the remaining part of this section we therefore study how a variable transformation affects the adjoint of an operator on an abstract level.

Lemma 32. *Let $\mathcal{D}(A)$ be dense in X and $A^* : X \supseteq \mathcal{D}(A^*) \to X$ denote the adjoint operator of A with respect to $(.\,,.)_T$. Then also $\mathcal{D}(B)$ is dense in X, and the adjoint of B with respect to $(.\,,.)_E$ is of the form $B^* = TA^*T^{-1}$ with $\mathcal{D}(B^*) = T\mathcal{D}(A^*)$.*

Proof. In our setting, $\|.\|_X$, $\|.\|_E$ and $\|.\|_T$ are equivalent. Yet, the following simple proof for the density of $\mathcal{D}(B)$ in $\big(X,\|.\|_E\big)$ provided $\mathcal{D}(A)$ is dense in $\big(X,\|.\|_T\big)$, even holds for unbounded T.

Let $\mathcal{D}(A)$ be dense in $\big(X,\|.\|_T\big)$ and $x \in X$ be arbitrary. Then there is a sequence $(y_n)_{n\in\mathbb{N}}$ in $\mathcal{D}(A)$, with $\lim_{n\to\infty}\|y_n - T^{-1}x\|_T = 0$. Hence, $\lim_{n\to\infty}\|Ty_n - x\|_E = \lim_{n\to\infty}\|y_n - T^{-1}x\|_T = 0$, which proves, that $\mathcal{D}(B) = T\mathcal{D}(A)$ is dense in $\big(X,\|.\|_E\big)$.

Furthermore, for $v \in X$ it holds

$$\begin{aligned} v \in \mathcal{D}(A^*) \quad &\Leftrightarrow \quad \exists y \in X: \quad (Au, v)_T = (u, y)_T\,, \quad u \in \mathcal{D}(A) \\ &\Leftrightarrow \quad \exists y \in X: \quad (TAT^{-1}Tu, Tv)_E = (Tu, Ty)_E\,, \quad u \in \mathcal{D}(A) \\ &\Leftrightarrow \quad \exists z \in X: \quad (Bw, Tv)_E = (w, z)_E\,, \quad w \in \mathcal{D}(B) \\ &\Leftrightarrow \quad Tv \in \mathcal{D}(B^*)\,. \end{aligned}$$

If one of these equivalent statements holds true, then y and z are unique with this property because of the density of $\mathcal{D}(A)$ and $\mathcal{D}(B)$ in X, and it is $y = A^*v$ and $z = B^*Tv$. So

$$B^*Tv = z = Ty = TA^*v = (TA^*T^{-1})Tv\,.$$

Together with the bijectivity of T, the last statement of this lemma follows. □

Lemma 33. *The operator A is symmetric with respect to $(.\,,.)_T$, iff B is symmetric with respect to $(.\,,.)_E$. Analogously, A is skew-symmetric with respect to $(.\,,.)_T$, iff B is skew-symmetric with respect to $(.\,,.)_E$.*

Proof. At first let A be symmetric with respect to $(.\,,.)_T$. Then

$$\begin{aligned} (Bv, w)_E &= (TAT^{-1}v, TT^{-1}w)_E = (AT^{-1}v, T^{-1}w)_T \\ &= (T^{-1}v, AT^{-1}w)_T = (v, TAT^{-1}w)_E = (v, Bw)_E\,, \end{aligned}$$

$v, w \in \mathcal{D}(B)$. The other direction follows from the symmetry of the similarity relation.

The second statement is proven analogously. □

4.2 A Special Class of Operators

The following assumption is motivated by our application. The operator A of section 4.1 will be of the form (2.14). Then B of section 4.1 will turn out to have the form $-P_1Q + P_2$ with two bounded linear operators $P_1, P_2 \in \mathcal{L}(X)$ with additional beneficial properties, which depend on the material parameters, and an unbounded operator $Q : X \supseteq \mathcal{D}(B) \to X$, which is independent of any material parameter. We will see then that the energy scalar product $(.\,,.)_E$ corresponding to the operator B is given by $(P_1^{-1}\,\cdot\,,\cdot)_X$.

In this short section we prove the well-definedness of such a scalar product under given assumptions on P_1 and also derive some norm estimates, which we will apply to norms related to the operators B and A in later sections.

Assumption 34. Throughout this section let $\big(X, (.\,,.)_X\big)$ be a non-trivial real Hilbert space and $\|.\|_X$ the norm induced by $(.\,,.)_X$.

Let $P_1, P_2, \tilde{P}_1 \in \mathcal{L}(X)$ and $Q : X \supseteq \mathcal{D}(Q) \to X$ be a linear not necessarily bounded operator with domain of definition $\mathcal{D}(Q)$. In addition let $P_1, \tilde{P}_1$ be boundedly invertible and $\tilde{P}_1$ self-adjoint and monotone, that is $(\tilde{P}_1 v, w)_X = (v, \tilde{P}_1 w)_X$ and $(\tilde{P}_1 w, w)_X \geq 0$, $v, w \in X$.

We use the notation $P \coloneqq (P_1, P_2)$ and introduce the linear operator $\beta(P) \coloneqq -P_1 Q + P_2 : X \supseteq \mathcal{D}(Q) \to X$. □

Lemma 35. *(Cauchy-Schwarz Inequality for Positive Semidefinite Symmetric Bilinear Forms)*

Let $(.\,,.)$ be a positive semidefinite symmetric bilinear form on X. Then

$$|(v,w)| \;\leq\; \sqrt{(v,v)}\sqrt{(w,w)}\,, \qquad\qquad v, w \in X\,. \tag{4.13}$$

Proof. Let $v, w \in X$. For every $\varepsilon > 0$ it follows from the positive semidefiniteness, bilinearity and symmetry of $(.\,,.)$, that

$$\begin{aligned} 0 \quad &\leq \quad \Big(v - \frac{(v,w)}{(w,w)+\varepsilon}\,w\,,\, v - \frac{(v,w)}{(w,w)+\varepsilon}\,w\Big) \\ &= \quad (v,v) \;-\; 2\,\frac{(v,w)^2}{(w,w)+\varepsilon} \;+\; \frac{(v,w)^2(w,w)}{\big((w,w)+\varepsilon\big)^2}\,, \end{aligned}$$

which is equivalent to

$$\sqrt{2 - \frac{(w,w)}{(w,w)+\varepsilon}}\;|(v,w)| \quad \leq \quad \sqrt{(v,v)}\,\sqrt{(w,w)+\varepsilon}\,.$$

For $\varepsilon \to 0$, the right-hand side of this inequality tends to the right-hand side of (4.13). If $(w,w) \neq 0$, also the left-hand side of this inequality tends to the left-hand side of (4.13), which is therefore proven in this case. If on the other hand $(w,w) = 0$, the left-hand side of the last inequality is equal to $\sqrt{2}\,|(v,w)|$, from where it follows that $(v,w) = 0$. So also in this case (4.13) is true. □

Lemma 36. *The scalar product $(v,w)_{\tilde{P}} \coloneqq (\tilde{P}_1^{-1} v, w)_X$, $v, w \in X$, is well-defined. For the norm $\|.\|_{\tilde{P}}$ induced by it, it holds*

$$\frac{1}{\sqrt{\|\tilde{P}_1\|_{\mathcal{L}(X,\|.\|_X)}}}\,\|w\|_X \quad \leq \quad \|w\|_{\tilde{P}} \quad \leq \quad \sqrt{\|\tilde{P}_1^{-1}\|_{\mathcal{L}(X,\|.\|_X)}}\,\|w\|_X\,, \tag{4.14}$$

$w \in X$. So $(.\,,.)_{\tilde{P}}$ is equivalent to $(.\,,.)_X$.

Furthermore, for the graph norms $\|\cdot\|_{Q,X} \coloneqq \|Q\,\cdot\,\|_X + \|\cdot\|_X$ *and* $\|\cdot\|_{\beta(P),\tilde{P}} \coloneqq \|\beta(P)\,\cdot\,\|_{\tilde{P}} + \|\cdot\|_{\tilde{P}}$ *it holds*

$$k_{P,\tilde{P}}\,\|w\|_{Q,X} \quad\leq\quad \|w\|_{\beta(P),\tilde{P}} \quad\leq\quad K_{P,\tilde{P}}\,\|w\|_{Q,X}\,, \tag{4.15}$$

$w \in \mathcal{D}(Q)$, *with*

$$\begin{aligned} k_{P,\tilde{P}} \quad &\coloneqq \quad \Bigg(\sqrt{\|\tilde{P}_1\|_{\mathcal{L}(X,\|.\|_X)}} \\ &\qquad \max\Big\{\|P_1^{-1}\|_{\mathcal{L}(X,\|.\|_X)}\,,\,\|P_1^{-1}\|_{\mathcal{L}(X,\|.\|_X)}\|P_2\|_{\mathcal{L}(X,\|.\|_X)}+1\Big\}\Bigg)^{-1}, \\ K_{P,\tilde{P}} \quad &\coloneqq \quad \sqrt{\|\tilde{P}_1^{-1}\|_{\mathcal{L}(X,\|.\|_X)}}\ \max\Big\{\|P_1\|_{\mathcal{L}(X,\|.\|_X)}\,,\,\|P_2\|_{\mathcal{L}(X,\|.\|_X)}+1\Big\}\,. \end{aligned}$$

For every $d > 0$ *there is* $c > 0$ *such that*

$$\frac{1}{c}\,\|w\|_{Q,X} \quad\leq\quad \|w\|_{\beta(P),\tilde{P}} \quad\leq\quad c\,\|w\|_{Q,X}\,, \tag{4.16}$$

$w \in \mathcal{D}(Q)$, *holds for all* $P_1, P_2, \tilde{P}_1$ *as in Assumption 34 with*

$$\|P_1\|_{\mathcal{L}(X,\|.\|_X)}\,,\ \|P_2\|_{\mathcal{L}(X,\|.\|_X)}\,,\ \|P_1^{-1}\|_{\mathcal{L}(X,\|.\|_X)}\,,\ \|\tilde{P}_1^{-1}\|_{\mathcal{L}(X,\|.\|_X)} \quad\leq\quad d\,.$$

Proof. Since $\tilde{P}_1$ is self-adjoint we have

$$\begin{aligned} (\tilde{P}_1^{-1}v, w)_X \;&=\; (\tilde{P}_1^{-1}v, \tilde{P}_1\tilde{P}_1^{-1}w)_X \;=\; (\tilde{P}_1\tilde{P}_1^{-1}v, \tilde{P}_1^{-1}w)_X \;=\; (v, \tilde{P}_1^{-1}w)_X \\ &=\; (\tilde{P}_1^{-1}w, v)_X\,, \end{aligned}$$

$v, w \in X$. So $(\tilde{P}_1^{-1}.\,,.)_X$ is symmetric.
As $\tilde{P}_1^{-1}$ is linear, $(\tilde{P}_1^{-1}.\,,.)_X$ is bilinear.
From the monotonicity of $\tilde{P}_1$ it follows

$$(\tilde{P}_1^{-1}w, w)_X \quad=\quad (\tilde{P}_1^{-1}w,\, \tilde{P}_1\tilde{P}_1^{-1}w)_X \quad\geq\quad 0\,, \qquad\qquad w \in X\,.$$

To prove the definiteness of $(\tilde{P}_1^{-1}.\,,.)_X$ we apply Lemma 35 to the positive semidefinite symmetric bilinear form $(\tilde{P}_1^{-1}.\,,.)_X$ and calculate

$$\|w\|_X^2 \quad=\quad \big(\tilde{P}_1^{-1}\tilde{P}_1 w, w\big)_X \quad\leq\quad \sqrt{\big(\tilde{P}_1^{-1}\tilde{P}_1 w, \tilde{P}_1 w\big)_X}\,\sqrt{\big(\tilde{P}_1^{-1}w, w\big)_X}\,, \tag{4.17}$$

$w \in X$, and by again using the Cauchy-Schwarz inequality we get

$$\begin{aligned} \big(\tilde{P}_1^{-1}\tilde{P}_1 w, \tilde{P}_1 w\big)_X \quad &= \quad (w,\, \tilde{P}_1 w)_X \quad\leq\quad \|w\|_X\,\|\tilde{P}_1 w\|_X \\ &\leq \quad \|\tilde{P}_1\|_{\mathcal{L}(X,\|.\|_X)}\,\|w\|_X^2\,, \end{aligned} \tag{4.18}$$

$w \in X$. By plugging (4.18) into (4.17) we get

$$\|w\|_X^2 \quad \leq \quad \sqrt{\|\tilde{P}_1\|_{\mathcal{L}(X,\|.\|_X)}}\,\|w\|_X \sqrt{(\tilde{P}_1^{-1}w, w)_X}\,, \qquad\qquad w \in X\,. \tag{4.19}$$

From this inequality it follows that whenever $(\tilde{P}_1^{-1}w, w)_X = 0$ for one $w \in X$ also $\|w\|_X^2 = 0$ and therefore $w = 0$. So $(\tilde{P}_1^{-1}.\,,.)_X$ actually is a scalar product.

The norm $\|.\|_{\tilde{P}}$ induced by $(\tilde{P}_1^{-1}.\,,.)_X$ has the form $\|w\|_{\tilde{P}} = \sqrt{(\tilde{P}_1^{-1}w, w)_X}$, $w \in X$. So we simply need to divide (4.19) by $\|w\|_X$ and $\sqrt{\|\tilde{P}_1\|_{\mathcal{L}(X,\|.\|_X)}} \neq 0$ to arrive at the left inequality of (4.14).

The right inequality of (4.14) follows with the Cauchy-Schwarz inequality from

$$\|w\|_{\tilde{P}}^2 \quad = \quad \big(\tilde{P}_1^{-1}w, w\big)_X \quad \leq \quad \|\tilde{P}_1^{-1}w\|_X\|w\|_X \quad \leq \quad \|\tilde{P}_1^{-1}\|_{\mathcal{L}(X,\|.\|_X)}\,\|w\|_X^2\,,$$

$w \in X$, by taking the root of both sides.

To prove (4.15), let $w \in X$. With (4.14) we get

$$\begin{aligned}
\|w\|_{Q,X} \quad &= \quad \|Qw\|_X \ + \ \|w\|_X \\
&= \quad \big\|P_1^{-1}\big[(-P_1Q + P_2)w - P_2w\big]\big\|_X \ + \ \|w\|_X \\
&\leq \quad \|P_1^{-1}\|_{\mathcal{L}(X,\|.\|_X)}\,\big\|\beta(P)w - P_2w\big\|_X \ + \ \|w\|_X \\
&\leq \quad \|P_1^{-1}\|_{\mathcal{L}(X,\|.\|_X)}\Big(\big\|\beta(P)w\big\|_X + \|P_2\|_{\mathcal{L}(X,\|.\|_X)}\|w\|_X\Big) \ + \ \|w\|_X \\
&\leq \quad \sqrt{\|\tilde{P}_1\|_{\mathcal{L}(X,\|.\|_X)}}\bigg(\|P_1^{-1}\|_{\mathcal{L}(X,\|.\|_X)}\Big(\big\|\beta(P)w\big\|_{\tilde{P}} \\
&\qquad\qquad + \|P_2\|_{\mathcal{L}(X,\|.\|_X)}\|w\|_{\tilde{P}}\Big) \ + \ \|w\|_{\tilde{P}}\bigg) \\
&\leq \quad \frac{1}{k_{P,\tilde{P}}}\|w\|_{\beta(P),\tilde{P}}\,.
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
&\|w\|_{\beta(P),\tilde{P}} \\
&\quad = \quad \big\|\beta(P)w\big\|_{\tilde{P}} + \|w\|_{\tilde{P}} \\
&\quad \leq \quad \sqrt{\|\tilde{P}_1^{-1}\|_{\mathcal{L}(X,\|.\|_X)}}\,\Big(\big\|\beta(P)w\big\|_X + \|w\|_X\Big) \\
&\quad = \quad \sqrt{\|\tilde{P}_1^{-1}\|_{\mathcal{L}(X,\|.\|_X)}}\,\Big(\big\|(-P_1Q + P_2)w\big\|_X + \|w\|_X\Big) \\
&\quad \leq \quad \sqrt{\|\tilde{P}_1^{-1}\|_{\mathcal{L}(X,\|.\|_X)}}\,\Big(\|P_1\|_{\mathcal{L}(X,\|.\|_X)}\|Qw\|_X \ + \ \|P_2\|_{\mathcal{L}(X,\|.\|_X)}\|w\|_X \ + \ \|w\|_X\Big) \\
&\quad \leq \quad K_{P,\tilde{P}}\,\|w\|_{Q,X}\,.
\end{aligned}$$

Finally, a direct estimate yields (4.16) for $c := \sqrt{d}\,(d+1)^2$. □

Chapter 5

Viscoelasticity: Unique Existence, Energy

In this chapter we apply the abstract theory developed in chapters 3 and 4 to the initial-boundary value problem given by the viscoelastic partial differential equation

$$
\begin{aligned}
\partial_t \mathbf{v}(x,t) \;&=\; \vartheta(x)\,\mathrm{div}\,\boldsymbol{\sigma}(x,t) \;+\; \tilde{\mathbf{f}}(x,t)\,,\\
\partial_t \boldsymbol{\sigma}(x,t) \;&=\; C\Big(\mu_H(x) + \sum_{l=1}^{L}\mu_{M,l}(x)\;,\;\kappa_H(x) + \sum_{l=1}^{L}\kappa_{M,l}(x)\Big)\,\varepsilon(\mathbf{v})(x,t)\\
&\qquad\qquad + \sum_{l=1}^{L}\boldsymbol{\eta}_l(x,t) \;+\; \mathbf{g}(x,t)\,, \qquad\qquad (5.1)\\
\partial_t \boldsymbol{\eta}_l(x,t) \;&=\; -\omega_{\boldsymbol{\sigma},l}(x) C\big(\mu_{M,l}(x)\,,\,\kappa_{M,l}(x)\big)\,\varepsilon(\mathbf{v})(x,t)\\
&\qquad\qquad -\; \omega_{\boldsymbol{\sigma},l}(x)\boldsymbol{\eta}_l(x,t)\,, \qquad\qquad l = 1,\ldots,L\,,
\end{aligned}
$$

$x \in D,\; t \in [0,t_1]$, introduced in (2.13), together with initial values

$$\mathbf{v}(x,0) \;=\; \mathbf{v}^{(0)}(x)\,, \qquad \boldsymbol{\sigma}(x,0) \;=\; \boldsymbol{\sigma}^{(0)}(x)\,, \qquad \boldsymbol{\eta}_l(x,0) \;=\; \boldsymbol{\eta}_l^{(0)}(x)\,, \quad l = 1,\ldots,L\,,$$

$x \in D$, and boundary values

$$\mathbf{v}(x,t) \;=\; \mathbf{0}\,,\; x \in \partial D_D\,,\; t \in [0,t_1]\,, \qquad \mathbf{n}(x)^\top \boldsymbol{\sigma}(x,t) \;=\; \mathbf{0}\,,\; x \in \partial D_N\,,\; t \in [0,t_1]\,.$$

Here, $D \subseteq \mathbb{R}^3$ denotes a non-empty open set with its boundary decomposed as $\partial D = \partial D_D \dot{\cup} \partial D_N$, $\mathbf{n}$ is the outer unit normal vector and $t_1 > 0$. We recall that $\tilde{\mathbf{f}} = \vartheta \mathbf{f}$.

According to section 2.6, we interpret problem (5.1) as an evolution equation

$$
\begin{aligned}
u'(t) \;&=\; -Au(t) \;+\; f(t)\,, \qquad t \in [0,t_1]\,,\\
u(0) \;&=\; u_0
\end{aligned}
\qquad (5.2)
$$

with $u = (\mathbf{v}, \boldsymbol{\sigma}, \boldsymbol{\eta})^\top$ and $\boldsymbol{\eta} = (\boldsymbol{\eta}_1, \dots, \boldsymbol{\eta}_L)^\top$, $u_0 = (\mathbf{v}^{(0)}, \boldsymbol{\sigma}^{(0)}, \boldsymbol{\eta}^{(0)})^\top$ and $\boldsymbol{\eta}^{(0)} = (\boldsymbol{\eta}_1^{(0)}, \dots, \boldsymbol{\eta}_L^{(0)})^\top$, $f = (\tilde{\mathbf{f}}, \mathbf{g}, \mathbf{0})^\top$, and the linear operator $A : X \supseteq \mathcal{D}(A) \to X$,

$$A \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma} \\ \boldsymbol{\eta}_1 \\ \vdots \\ \boldsymbol{\eta}_L \end{pmatrix} = - \begin{pmatrix} \vartheta \operatorname{div} \boldsymbol{\sigma} \\ C\Big(\mu_H + \sum_{l=1}^L \mu_{M,l}\,,\ \kappa_H + \sum_{l=1}^L \kappa_{M,l}\Big)\, \varepsilon(\mathbf{v}) \ +\ \sum_{l=1}^L \boldsymbol{\eta}_l \\ -\omega_{\boldsymbol{\sigma},1} C\big(\mu_{M,1}\,,\ \kappa_{M,1}\big)\, \varepsilon(\mathbf{v}) \ -\ \omega_{\boldsymbol{\sigma},1} \boldsymbol{\eta}_1 \\ \vdots \\ -\omega_{\boldsymbol{\sigma},L} C\big(\mu_{M,L}\,,\ \kappa_{M,L}\big)\, \varepsilon(\mathbf{v}) \ -\ \omega_{\boldsymbol{\sigma},L} \boldsymbol{\eta}_L \end{pmatrix}, \quad (5.3)$$

$(\mathbf{v}, \boldsymbol{\sigma}, \boldsymbol{\eta})^\top \in \mathcal{D}(A)$, on a suitable Hilbert space X and domain of definition $\mathcal{D}(A)$.

The purpose of the next section is to define these spaces.

5.1 Function Spaces

Notation 37. We assume $D \subseteq \mathbb{R}^3$ to be any non-empty open set. Furthermore let $\partial D_D \subseteq \partial D$ be an arbitrary subset of its boundary ∂D and $\partial D_N := \partial D \setminus \partial D_D$.

On the vector space $\mathbb{R}^3$ we make use of the canonical scalar product

$$\mathbf{a} \cdot \mathbf{b} \quad := \quad \sum_{i=1}^{3} a_i b_i\,, \qquad \mathbf{a} = (a_i)_{i=1,2,3}\,, \quad \mathbf{b} = (b_i)_{i=1,2,3} \quad \in \quad \mathbb{R}^3\,.$$

On the spaces $\mathbb{R}^{3\times3}$ and $\mathbb{R}^{3\times3}_{\mathrm{sym}}$ we use the Frobenius inner product which is defined as

$$\mathbf{M} : \mathbf{N} \ := \ \sum_{i,j=1}^{3} m_{ij} n_{ij}\,, \qquad \mathbf{M} = (m_{ij})_{i,j=1,2,3}\,, \quad \mathbf{N} = (n_{ij})_{i,j=1,2,3} \quad \in \quad \mathbb{R}^{3\times3}\,.$$

Having these scalar products we can define the vector spaces

$$L^2(D, \mathbb{R}^3) \ := \ \Big\{ \mathbf{v} : D \to \mathbb{R}^3 : \quad \mathbf{v} \text{ is measurable and } \int_D \mathbf{v}(x) \cdot \mathbf{v}(x)\, dx \ < \ \infty\,. \Big\}$$

and

$$L^2(D, \mathbb{R}^{3\times3}_{\mathrm{sym}}) := \Big\{ \boldsymbol{\sigma} : D \to \mathbb{R}^{3\times3}_{\mathrm{sym}} : \ \boldsymbol{\sigma} \text{ is measurable and } \int_D \boldsymbol{\sigma}(x) : \boldsymbol{\sigma}(x)\, dx \ < \ \infty\,. \Big\}$$

together with scalar products

$$\big(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\big)_{L^2(D,\mathbb{R}^3)} \quad := \quad \int_D \mathbf{v}^{(1)}(x) \cdot \mathbf{v}^{(2)}(x)\, dx\,, \qquad \mathbf{v}^{(1)}, \mathbf{v}^{(2)} \in L^2(D, \mathbb{R}^3)$$

and

$$\left(\boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)}\right)_{L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}})} \quad := \quad \int_D \boldsymbol{\sigma}^{(1)}(x) : \boldsymbol{\sigma}^{(2)}(x)\, dx\,, \qquad \boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)} \in L^2(D, \mathbb{R}^{3\times 3}_{\mathrm{sym}})$$

respectively.

Now for the Hilbert space on which A in (5.3) acts we choose

$$X \quad := \quad L^2(D,\mathbb{R}^3) \times L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}}) \times L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}})^L \tag{5.4}$$

with

$$(u_1, u_2)_X \;:=\; \left(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\right)_{L^2(D,\mathbb{R}^3)} + \left(\boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)}\right)_{L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}})} + \sum_{l=1}^{L} \left(\boldsymbol{\eta}_l^{(1)}, \boldsymbol{\eta}_l^{(2)}\right)_{L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}})}\,, \tag{5.5}$$

$u_1 = (\mathbf{v}^{(1)}, \boldsymbol{\sigma}^{(1)}, \boldsymbol{\eta}^{(1)})$, $u_2 = (\mathbf{v}^{(2)}, \boldsymbol{\sigma}^{(2)}, \boldsymbol{\eta}^{(2)}) \in X$. The norm on X induced by the scalar product $(\cdot\,,\cdot)_X$ is denoted by $\|.\|_X$.

Also we make use of spaces of k-times continuously differentiable functions with compact support

$$\begin{aligned} C^k_c(\Omega,\, \mathbb{R}^3) \quad &:= \quad \left\{\boldsymbol{\varphi} \in C^k(\Omega, \mathbb{R}^3) : \quad \mathrm{supp}(\boldsymbol{\varphi}) \subseteq \Omega \text{ is compact.}\right\}, \\ C^k_c(\Omega,\, \mathbb{R}^{3\times 3}_{\mathrm{sym}}) \quad &:= \quad \left\{\boldsymbol{\psi} \in C^k(\Omega, \mathbb{R}^{3\times 3}_{\mathrm{sym}}) : \quad \mathrm{supp}(\boldsymbol{\psi}) \subseteq \Omega \text{ is compact.}\right\}, \end{aligned}$$

where $\Omega \subseteq \mathbb{R}^3$ is any open set, $k \in \mathbb{N}_0 \cup \{\infty\}$ and $C^0(\Omega, \dots)$ stands for merely continuous functions. □

Lemma 38.

(a) The space $C^\infty_c(D,\mathbb{R}^3)$ is dense in $L^2(D,\mathbb{R}^3)$.

(b) The space $C^\infty_c(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}})$ is dense in $L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}})$.

Proof. We demonstrate the proof for (b). Let $\boldsymbol{\sigma} = (\sigma_{ij})_{i,j=1,2,3} \in L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}})$. Then $\sigma_{ij} \in L^2(D,\mathbb{R})$, $i,j = 1,2,3$. By the analogous result for scalar valued functions which is a special case of Corollary 3.5 in [15] there is $\left(\sigma_{ij}^{(n)}\right)_{n\in\mathbb{N}} \in C^\infty_c(D,\mathbb{R})^{\mathbb{N}}$ with $\lim_{n\to\infty} \|\sigma_{ij}^{(n)} - \sigma_{ij}\|_{L^2(D,\mathbb{R})} = 0$ for each $(i,j) \in \{1,2,3\}^2$ with $j \geq i$. With $\sigma_{ij}^{(n)} := \sigma_{ji}^{(n)}$, $n \in \mathbb{N}$ for $j < i$ and $\boldsymbol{\sigma}^{(n)} := (\sigma_{ij}^{(n)})_{i,j=1,2,3}$, $n \in \mathbb{N}$, it holds $\boldsymbol{\sigma}^{(n)} \in C^\infty_c(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}})$, $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \|\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}\|_{L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}})} \quad = \quad \lim_{n\to\infty} \sqrt{\sum_{i,j=1}^{3} \|\sigma_{ij}^{(n)} - \sigma_{ij}\|^2_{L^2(D,\mathbb{R})}} \quad = \quad 0\,.$$

In the same way, (a) can be shown. □

Notation 39. For $\mathbf{v} = (v_i)_{i=1,2,3} \in C_c^1(\Omega, \mathbb{R}^3)$ and $\boldsymbol{\sigma} \in C_c^1(\Omega, \mathbb{R}^{3\times3}_{\text{sym}})$ we define $\varepsilon(\mathbf{v})$, $\operatorname{div}\mathbf{v}$ and $\operatorname{div}\boldsymbol{\sigma}$ as done in section 2.4 by $\varepsilon(\mathbf{v}) := \big(D\mathbf{v} + (D\mathbf{v})^\top\big)/2$ with the Jacobian matrix $D\mathbf{v}$, $\operatorname{div}\mathbf{v} = \sum_{i=1}^3 \partial_i v_i$ and $\operatorname{div}\boldsymbol{\sigma} = \sum_{j=1}^3 \partial_j \boldsymbol{\sigma}_{*j}$. □

Lemma 40. *(Partial Integration)*

Let $\Omega \subseteq \mathbb{R}^3$ be a Lipschitz domain, $\mathbf{n}$ its exterior unit normal vector field, $\mathbf{v} \in C_c^1(\mathbb{R}^3, \mathbb{R}^3)$ and $\boldsymbol{\sigma} \in C_c^1(\mathbb{R}^3, \mathbb{R}^{3\times3}_{\text{sym}})$. Then

$$\int_\Omega \boldsymbol{\sigma} : \varepsilon(\mathbf{v}) \; + \; \operatorname{div}(\boldsymbol{\sigma}) \cdot \mathbf{v} \, dx \quad = \quad \int_{\partial\Omega} (\mathbf{n}^\top \boldsymbol{\sigma}) \, \mathbf{v} \, ds \,, \tag{5.6}$$

where ds indicates integration with respect to the two-dimensional surface measure.

Proof. For $\mathbf{v} = (v_i)_{i=1,2,3}$, $\boldsymbol{\sigma} = (\sigma_{ij})_{i,j=1,2,3}$ and $\operatorname{div}\boldsymbol{\sigma} = (d_i)_{i=1,2,3}$ we use the symmetry of $\boldsymbol{\sigma}$ to calculate

$$\begin{aligned}
\operatorname{div}(\boldsymbol{\sigma}\mathbf{v}) \quad &= \quad \sum_{j=1}^3 \partial_i \Big(\sum_{j=1}^3 \sigma_{ij} v_j \Big) \\
&= \quad \sum_{i,j=1}^3 \sigma_{ij} (\partial_i v_j) \; + \; \sum_{i,j=1}^3 (\partial_i \sigma_{ij}) v_j \\
&= \quad \frac{1}{2} \Big(\sum_{i,j=1}^3 \sigma_{ij} (\partial_i v_j) \; + \; \sum_{i,j=1}^3 \sigma_{ji} (\partial_i v_j) \Big) \; + \; \sum_{i,j=1}^3 (\partial_i \sigma_{ji}) v_j \\
&= \quad \sum_{i,j=1}^3 \sigma_{ij} \, \frac{\partial_i v_j + \partial_j v_i}{2} \; + \; \sum_{j=1}^3 d_j v_j \\
&= \quad \boldsymbol{\sigma} : \varepsilon(\mathbf{v}) \; + \; \operatorname{div}(\boldsymbol{\sigma}) \cdot \mathbf{v} \,.
\end{aligned}$$

On the other hand with $\mathbf{n} = (n_i)_{i=1,2,3}$ it is

$$(\boldsymbol{\sigma}\mathbf{v}) \cdot \mathbf{n} \quad = \quad \sum_{i,j=1}^3 \sigma_{ij} v_j n_i \quad = \quad (\mathbf{n}^\top \boldsymbol{\sigma}) \, \mathbf{v} \,.$$

Now (5.6) follows from the Gauß divergence theorem, which can be found as Theorem 3.34 in [15]. □

Notation 41. Motivated by (5.6) we can define ε and div in a weak sense in the usual way.

More precisely, for any $\mathbf{v} \in L^2(D, \mathbb{R}^3)$ we say, $\varepsilon(\mathbf{v})$ exists in the weak sense, iff there is $\mathbf{g} \in L^2(D, \mathbb{R}^{3\times3}_{\text{sym}})$ such that

$$\int_D \mathbf{v} \cdot \operatorname{div}(\Psi) \, dx \quad = \quad -\int_D \mathbf{g} : \Psi \, dx \,, \qquad\qquad \Psi \in C_c^\infty(D, \mathbb{R}^{3\times3}_{\text{sym}}) \,. \tag{5.7}$$

If it exists, $\mathbf{g}$ is unique with this property because by Lemma 38, the space $C_c^\infty(D, \mathbb{R}^{3\times3}_{\mathrm{sym}})$ is dense in $L^2(D, \mathbb{R}^{3\times3}_{\mathrm{sym}})$. In this case, $\mathbf{g}$ is denoted by $\varepsilon(\mathbf{v})$.

In this way we introduce the set

$$H(\varepsilon, D, \mathbb{R}^3) \quad := \quad \Big\{ \mathbf{v} \in L^2(D, \mathbb{R}^3) \ : \ \varepsilon(\mathbf{v}) \text{ exists in the weak sense.} \Big\}.$$

It forms a vector space and can be equipped with the scalar product

$$\big(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\big)_V \quad := \quad \big(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\big)_{L^2(D,\mathbb{R}^3)} + \big(\varepsilon(\mathbf{v}^{(1)}), \varepsilon(\mathbf{v}^{(2)})\big)_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})}, \tag{5.8}$$

$\mathbf{v}^{(1)}, \mathbf{v}^{(2)} \in H(\varepsilon, D, \mathbb{R}^3)$. The norm induced by it is denoted by $\|.\|_V$.

In the same way for any $\boldsymbol{\sigma} \in L^2(D, \mathbb{R}^{3\times3}_{\mathrm{sym}})$ we say, $\operatorname{div} \boldsymbol{\sigma}$ exists in the weak sense, iff there is $\mathbf{h} \in L^2(D, \mathbb{R}^3)$ such that

$$\int_D \boldsymbol{\sigma} : \varepsilon(\boldsymbol{\varphi})\, dx \quad = \quad - \int_D \mathbf{h} \cdot \boldsymbol{\varphi}\, dx\,, \qquad\qquad \boldsymbol{\varphi} \in C_c^\infty(D, \mathbb{R}^3)\,.$$

In this case $\mathbf{h}$ is unique, since $C_c^\infty(D, \mathbb{R}^3)$ is dense in $L^2(D, \mathbb{R}^3)$ due to Lemma 38, and we denote $\mathbf{h}$ by $\operatorname{div} \boldsymbol{\sigma}$.

In this way we define the vector space

$$H(\operatorname{div}, D, \mathbb{R}^{3\times3}_{\mathrm{sym}}) \quad := \quad \Big\{ \boldsymbol{\sigma} \in L^2(D, \mathbb{R}^{3\times3}_{\mathrm{sym}}) \ : \ \operatorname{div} \boldsymbol{\sigma} \text{ exists in the weak sense.} \Big\}.$$

It can be equipped with the scalar product

$$\big(\boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)}\big)_S \quad := \quad \big(\boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)}\big)_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})} + \big(\operatorname{div} \boldsymbol{\sigma}^{(1)}, \operatorname{div} \boldsymbol{\sigma}^{(2)}\big)_{L^2(D,\mathbb{R}^3)},$$

$\boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)} \in H(\operatorname{div}, D, \mathbb{R}^{3\times3}_{\mathrm{sym}})$. The norm induced by $(.\,,.)_S$ is denoted by $\|.\|_S$. $\square$

Lemma 42. *The spaces* $\big(H(\varepsilon, D, \mathbb{R}^3), (.\,,.)_V\big)$ *and* $\big(H(\operatorname{div}, D, \mathbb{R}^{3\times3}_{\mathrm{sym}}), (.\,,.)_S\big)$ *are Hilbert spaces.*

Proof. Let $(\mathbf{v}_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\big(H(\varepsilon, D, \mathbb{R}^3), (.\,,.)_V\big)$. Since

$$\begin{aligned} \|\mathbf{v}_n - \mathbf{v}_m\|^2_{L^2(D,\mathbb{R}^3)}\,, \quad & \big\|\varepsilon(\mathbf{v}_n - \mathbf{v}_m)\big\|^2_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})} \\ & \leq \quad \|\mathbf{v}_n - \mathbf{v}_m\|^2_{L^2(D,\mathbb{R}^3)} + \big\|\varepsilon(\mathbf{v}_n - \mathbf{v}_m)\big\|^2_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})} \\ & = \quad \|\mathbf{v}_n - \mathbf{v}_m\|^2_V\,, \end{aligned}$$

$n, m \in \mathbb{N}$, the sequences $(\mathbf{v}_n)_{n\in\mathbb{N}}$ and $\big(\varepsilon(\mathbf{v}_n)\big)_{n\in\mathbb{N}}$ are also Cauchy with respect to $\|.\|_{L^2(D,\mathbb{R}^3)}$ and $\|.\|_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})}$, respectively. Since $\big(L^2(D, \mathbb{R}^3), \|.\|_{L^2(D,\mathbb{R}^3)}\big)$ and $\big(L^2(D, \mathbb{R}^{3\times3}_{\mathrm{sym}}), \|.\|_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})}\big)$ are complete, there are both $\mathbf{v} \in L^2(D, \mathbb{R}^3)$ and $\mathbf{g} \in$

$L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})$ such that $\|\mathbf{v}_n-\mathbf{v}\|_{L^2(D,\mathbb{R}^3)}\to 0$, $n\to\infty$ and $\big\|\varepsilon(\mathbf{v}_n)-\mathbf{g}\big\|_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})}\to 0$, $n\to\infty$. Now for any fixed $\boldsymbol{\psi}\in C_c^\infty(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})$ we can apply the Cauchy-Schwarz inequality to prove

$$\begin{aligned}\int_D \mathbf{v}\cdot\,\mathrm{div}\,(\Psi)\;+\;\mathbf{g}:\Psi\;dx \quad &= \quad \int_D (\lim_{n\to\infty}\mathbf{v}_n)\cdot\,\mathrm{div}\,(\Psi)\;+\;\big(\lim_{n\to\infty}\varepsilon(\mathbf{v}_n)\big):\Psi\;dx\\ &= \quad \lim_{n\to\infty}\int_D \mathbf{v}_n\cdot\,\mathrm{div}\,(\Psi)\;+\;\varepsilon(\mathbf{v}_n):\Psi\;dx\\ &= \quad 0\,.\end{aligned}$$

So by definition, $\mathbf{v}\in H(\varepsilon,D,\mathbb{R}^3)$, and $\big(H(\varepsilon,D,\,\mathbb{R}^3),(\,.\,,.)_V\big)$ is complete.

The same argument applies to $\big(H(\,\mathrm{div}\,,D,\,\mathbb{R}^{3\times3}_{\mathrm{sym}}),(\,.\,,.)_S\big)$. □

Notation 43. Let

$$V \quad := \quad \overline{\Big\{\,\boldsymbol{\varphi}\in C^\infty(D,\mathbb{R}^3)\cap H(\varepsilon,D,\mathbb{R}^3)\;:\quad \partial D_D\subseteq\mathbb{R}^3\setminus\mathrm{supp}(\boldsymbol{\varphi})\,\Big\}}^{\|.\|_V}\;, \tag{5.9}$$

where the bar stands for closure in $\big(H(\varepsilon,D,\mathbb{R}^3),\;\|.\|_V\big)$. Furthermore, let

$$S \;:=\; \Big\{\,\boldsymbol{\sigma}\in H(\,\mathrm{div}\,,D,\mathbb{R}^{3\times3}_{\mathrm{sym}})\;:\;\forall\boldsymbol{\varphi}\in V:\int_D\varepsilon(\boldsymbol{\varphi}):\boldsymbol{\sigma}\;+\;\boldsymbol{\varphi}\cdot\,\mathrm{div}\,\boldsymbol{\sigma}\,dx=0\,\Big\}\,. \tag{5.10}$$

Now, for the domain of definition of A in (5.3) we choose

$$\mathcal{D}(A)\quad:=\quad V\times S\times L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})^L\,. \tag{5.11}$$

This linear subspace of X can be equipped with the graph norm

$$\|u\|_{A,X}\quad:=\quad\|u\|_X\;+\;\|Au\|_X\,,\qquad\qquad u\in\mathcal{D}(A) \tag{5.12}$$

of A. □

Lemma 44. *The space V together with the scalar product $(\,.\,,.)_V$ is a Hilbert space.*

Proof. The set shown in (5.9) which the closure is taken of, is a vector space. Because for two elements $\mathbf{u}$ and $\mathbf{v}$ and a scalar $\alpha\in\mathbb{R}$ it holds

$$\begin{aligned}\mathrm{supp}\big(\alpha\mathbf{u}+\mathbf{v}\big)\quad&\subseteq\quad\mathrm{supp}(\alpha\mathbf{u})\,\cup\,\mathrm{supp}(\mathbf{v})\quad\subseteq\quad\mathrm{supp}(\mathbf{u})\,\cup\,\mathrm{supp}(\mathbf{v})\\ &\subseteq\quad\mathbb{R}^3\setminus\partial D_D\,.\end{aligned}$$

So also $\alpha\mathbf{u}+\mathbf{v}$ is contained. Furthermore $\mathbf{0}$ is an element. Thus it is nonempty.

The closure of a linear subspace of any normed space is a vector space again. Finally, by construction and Lemma 42, V is a closed subspace of the complete space $\big(H(\varepsilon,D,\mathbb{R}^3),\;\|.\|_V\big)$. So it is a Hilbert space on its own. □

Lemma 45. *The space $\big(S, (.\,,.)_S\big)$ is a Hilbert space.*

Proof. For fixed $\boldsymbol{\varphi} \in V$ let

$$\ell_{\boldsymbol{\varphi}} : \quad H(\operatorname{div}, D, \mathbb{R}^{3\times 3}_{\mathrm{sym}}) \to \mathbb{R}, \qquad \boldsymbol{\sigma} \mapsto \int_D \varepsilon(\boldsymbol{\varphi}) : \boldsymbol{\sigma} + \boldsymbol{\varphi} \cdot \operatorname{div} \boldsymbol{\sigma} \, dx .$$

Then $\ell_{\boldsymbol{\varphi}}$ obviously is linear. Furthermore it is bounded with respect to $\|.\|_S$, because with the Cauchy-Schwarz inequality it follows

$$\begin{aligned} \big|\ell_{\boldsymbol{\varphi}}(\boldsymbol{\sigma})\big| &\le \int_D \big|\varepsilon(\boldsymbol{\varphi}) : \boldsymbol{\sigma}\big| \, dx + \int_D \big|\boldsymbol{\varphi} \cdot \operatorname{div} \boldsymbol{\sigma}\big| \, dx \\ &\le \|\varepsilon(\boldsymbol{\varphi})\|_{L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}})} \|\boldsymbol{\sigma}\|_{L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}})} + \|\boldsymbol{\varphi}\|_{L^2(D,\mathbb{R}^3)} \|\operatorname{div} \boldsymbol{\sigma}\|_{L^2(D,\mathbb{R}^3)} \\ &\le \sqrt{\|\boldsymbol{\varphi}\|^2_{L^2(D,\mathbb{R}^3)} + \|\varepsilon(\boldsymbol{\varphi})\|^2_{L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}})}} \sqrt{\|\boldsymbol{\sigma}\|^2_{L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}})} + \|\operatorname{div} \boldsymbol{\sigma}\|^2_{L^2(D,\mathbb{R}^3)}} \\ &= \|\boldsymbol{\varphi}\|_V \, \|\boldsymbol{\sigma}\|_S , \end{aligned}$$

$\boldsymbol{\sigma} \in H(\operatorname{div}, D, \mathbb{R}^{3\times 3}_{\mathrm{sym}})$.

Now

$$S = \bigcap_{\boldsymbol{\varphi} \in V} \ell_{\boldsymbol{\varphi}}^{-1}\big(\{0\}\big) ,$$

which is an intersection of closed normed subspaces of the complete normed space $\big(H(\operatorname{div}, D, \mathbb{R}^{3\times 3}_{\mathrm{sym}}), \|.\|_S\big)$ and therefore a closed and hence complete normed subspace itself. □

Lemma 46. *The space V is dense in $L^2(D, \mathbb{R}^3)$.*

Proof. Since

$$C_c^\infty(D, \mathbb{R}^3) \subseteq \Big\{ \boldsymbol{\varphi} \in C^\infty(D, \mathbb{R}^3) \cap H(\varepsilon, D, \mathbb{R}^3) : \partial D_D \subseteq \mathbb{R}^3 \setminus \operatorname{supp}(\boldsymbol{\varphi}) \Big\}$$

this follows from Lemma 38(a). □

In the same way, S is dense in $L^2(D, \mathbb{R}^{3\times 3}_{\mathrm{sym}})$. To prove this we could use Lemma 38 together with Lemma 40. Instead we postpone this result to Lemma 76 where we will have an elegant method at hand to prove it.

Remark 47. *With a look at* (5.6) *and the property $\mathbf{v}|_{\partial D_D} = \mathbf{0}$ of every $\mathbf{v}$ in the dense subspace $\big\{\boldsymbol{\varphi} \in C^\infty(D, \mathbb{R}^3) \cap H(\varepsilon, D, \mathbb{R}^3) : \partial D_D \subseteq \mathbb{R}^3 \setminus \operatorname{supp}(\boldsymbol{\varphi})\big\}$ of V used in the definition* (5.9)*, we see that the defining condition on the elements $\boldsymbol{\sigma}$ of the subspace S of $H(\operatorname{div}, D, \mathbb{R}^{3\times 3}_{\mathrm{sym}})$ formulated in* (5.10) *is a variational interpretation of the boundary condition $\mathbf{n}^\top \boldsymbol{\sigma}|_{\partial D_N} = \mathbf{0}$.*

It is worth noting that besides D being an open set, this construction does not need any further assumptions on the regularity of D, the boundary ∂D of D as well as the subsets ∂D_D and ∂D_N of ∂D. Of course the degenerate case $D = \emptyset$ does not matter in applications. □

Remark 48. *We further mention the space* $H^1(D,\mathbb{R}^3)$ *consisting of all* $\mathbf{v} \in L^2(D,\mathbb{R}^3)$ *for which there is* $\mathbf{g} \in L^2(D,\mathbb{R}^{3\times 3})$ *such that*

$$\int_D \mathbf{v}\,(\nabla\varphi)^\top\,ds \quad = \quad -\int_D \mathbf{g}\,\varphi\,dx\,, \qquad\qquad \varphi \in C_c^\infty(D,\mathbb{R})\,.$$

If it exists, $\mathbf{g}$ *is denoted by* $D\mathbf{v}$*. The vector space* $H^1(D,\mathbb{R}^3)$ *is endowed with the norm*

$$\|\mathbf{v}\|_{H^1(D,\mathbb{R}^3)} \quad := \quad \sqrt{\|\mathbf{v}\|^2_{L^2(D,\mathbb{R}^3)} \,+\, \|D\mathbf{v}\|^2_{L^2(D,\mathbb{R}^{3\times 3})}}\,,$$

$\mathbf{v} \in H^1(D,\mathbb{R}^3)$.

In the special case where D *is a bounded, connected Lipschitz domain, it is stated in [7] (pp. 291-292) that*

$$H(\varepsilon,D,\mathbb{R}^3) \quad = \quad H^1(D,\mathbb{R}^3)\,.$$

To prove this theorem one shows that $\|.\|_{H(\varepsilon,D,\mathbb{R}^3)}$ *and* $\|.\|_{H^1(D,\mathbb{R}^3)}$ *are equivalent, where one side is easily verified via the calculation*

$$\begin{aligned}\|\varepsilon(\mathbf{v})\|^2_{L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}})} \;&=\; \int_D \sum_{i,j=1}^{3} \frac{(\partial_i v_j + \partial_j v_i)^2}{4}\,dx \;\le\; \int_D \sum_{i,j=1}^{3} \frac{(\partial_i v_j)^2 + (\partial_j v_i)^2}{2}\,dx \\ &=\; \int_D \sum_{i,j=1}^{3} (\partial_i v_j)^2\,dx \;=\; \|D\mathbf{v}\|^2_{L^2(D,\mathbb{R}^{3\times 3})}\,,\end{aligned}$$

$\mathbf{v} = (v_1,v_2,v_3) \in H^1(D,\mathbb{R}^3) \subseteq H(\varepsilon,D,\mathbb{R}^3)$*. The other follows from Korn's inequality, a proof of which can be found in [16] and which states that there is* $c > 0$ *such that also*

$$\|D\mathbf{v}\|_{L^2(D,\mathbb{R}^{3\times 3})} \quad \le \quad c\,\|\varepsilon(\mathbf{v})\|_{L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}})}\,,$$

$\mathbf{v} \in H^1(D,\mathbb{R}^3)$. □

Finally, we choose spaces for the parameters in (5.3).

Assumption 49. We assume all material parameters to be positive, measurable real-valued functions on D which are essentially bounded from above and below, i.e.

$$\begin{aligned}&\vartheta,\ \mu_H,\ \mu_{M,1},\ldots,\mu_{M,L},\ \kappa_H,\ \kappa_{M,1},\ldots,\kappa_{M,L},\ \omega_{\boldsymbol{\sigma},1},\ldots,\omega_{\boldsymbol{\sigma},L} \\ &\qquad\in\; L^\infty_+(D) \;:=\; \left\{\alpha \in L^\infty(D)\;:\;\alpha > 0,\ \frac{1}{\alpha} \in L^\infty(D)\right\}.\end{aligned} \tag{5.13}$$

For better readability we introduce the notation

$$\begin{aligned} \vartheta_0 &:= \frac{1}{\|\frac{1}{\vartheta}\|_{L^\infty(D)}}\,, & \mu_{H,0} &:= \frac{1}{\|\frac{1}{\mu_H}\|_{L^\infty(D)}}\,, & \kappa_{H,0} &:= \frac{1}{\|\frac{1}{\kappa_H}\|_{L^\infty(D)}}\,, \\ \mu_{M,l,0} &:= \frac{1}{\|\frac{1}{\mu_{M,l}}\|_{L^\infty(D)}}\,, & \kappa_{M,l,0} &:= \frac{1}{\|\frac{1}{\kappa_{M,l}}\|_{L^\infty(D)}}\,, & \omega_{\boldsymbol{\sigma},l,0} &:= \frac{1}{\|\frac{1}{\omega_{\boldsymbol{\sigma},l}}\|_{L^\infty(D)}}\,, \end{aligned}$$

$l = 1, \dots, L$. Then

$$\begin{aligned} \vartheta(x) &\geq \vartheta_0 > 0\,, & \mu_H(x) &\geq \mu_{H,0} > 0\,, & \kappa_H(x) &\geq \kappa_{H,0} > 0\,, \\ \mu_{M,l}(x) &\geq \mu_{M,l,0} > 0\,, & \kappa_{M,l}(x) &\geq \kappa_{M,l,0} > 0\,, & \omega_{\boldsymbol{\sigma},l}(x) &\geq \omega_{\boldsymbol{\sigma},l,0} > 0\,, \end{aligned}$$

for almost all $x \in D$ and all $l \in \{1, \dots, L\}$.

Furthermore, we consider an initial value $u_0 = (\mathbf{v}^{(0)}, \boldsymbol{\sigma}^{(0)}, \boldsymbol{\eta}^{(0)})^\top \in \mathcal{D}(A)$ with $\mathcal{D}(A)$ as in (5.11), and inhomogeneities $\mathbf{f} \in W^{1,1}\big((0,t_1), L^2(D,\mathbb{R}^3)\big)$ and $\mathbf{g} \in W^{1,1}\big((0,t_1), L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}})\big)$. Then also $\tilde{\mathbf{f}} = \vartheta \mathbf{f} \in W^{1,1}\big((0,t_1), L^2(D,\mathbb{R}^3)\big)$. $\square$

5.2 The Elastic Stiffness Tensor

Many of the following calculations will involve the elastic stiffness tensors, which we defined by

$$C(m,k):\ \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}\,, \qquad C(m,k)\mathbf{M} = m\mathbf{M} + \frac{k-m}{3}\,\mathrm{trace}\,(\mathbf{M})\mathbf{I} \tag{5.14}$$

for $m, k \in \mathbb{R}$ in (2.9), where $\mathbf{I} \in \mathbb{R}^{3\times 3}$ denotes the unit matrix. Therefore we need some algebraic properties as well as estimates of these maps.

First we recall that

$$C(m,k)\mathbf{M} \quad = \quad m\,\mathrm{dev}\,\mathbf{M} \;+\; k\,\frac{\mathrm{trace}\,(\mathbf{M})}{3}\,\mathbf{I}\,,$$

$m, k \in \mathbb{R}$, $\mathbf{M} \in \mathbb{R}^{3\times 3}$, where the deviatoric part $\mathrm{dev}\,\mathbf{M} = \mathbf{M} - (\,\mathrm{trace}\,(\mathbf{M})/3)\,\mathbf{I}$ of $\mathbf{M} \in \mathbb{R}^{3\times 3}$ has been introduced in (2.8).

Lemma 50. *The set $\{C(m,k):\ m, k \in \mathbb{R}\}$ forms a commutative subalgebra with unity of $\mathcal{L}(\mathbb{R}^{3\times 3})$ and $\mathcal{L}(\mathbb{R}^{3\times 3}_{\mathrm{sym}})$, respectively, and the maps*

$$\begin{aligned} \iota_1 :\quad \mathbb{R}^2 &\to \mathcal{L}(\mathbb{R}^{3\times 3})\,, & (m,k) &\mapsto C(m,k)\,, \\ \iota_2 :\quad \mathbb{R}^2 &\to \mathcal{L}(\mathbb{R}^{3\times 3}_{\mathrm{sym}})\,, & (m,k) &\mapsto C(m,k) \end{aligned} \tag{5.15}$$

are injective algebra homomorphisms. This follows from the following properties. For $m, m_1, m_2, k, k_1, k_2, \lambda \in \mathbb{R}$, $\mathbf{M}, \mathbf{N} \in \mathbb{R}^{3\times 3}$ *it is* *(a)* $\qquad C(m_1,k_1)+C(m_2,k_2) \quad = \quad C(m_1+m_2\,,\,k_1+k_2)\,,$

(b) $\qquad C(0,0) \quad = \quad 0_{\mathcal{L}(\mathbb{R}^{3\times3})}\,,$

(c) $\qquad C(m_1,k_1)C(m_2,k_2) \quad = \quad C(m_1m_2\,,\,k_1k_2)\,,$

(d) $\qquad C(1,1) \quad = \quad \mathrm{Id}_{\mathcal{L}(\mathbb{R}^{3\times3})}\,,$

(e) $\qquad \lambda C(m,k) \quad = \quad C(\lambda m\,,\,\lambda k)\,,$

(f) $\qquad C(m_1,k_1)C(m_2,k_2) \quad = \quad C(m_2,k_2)C(m_1,k_1)\,,$

(g) $\qquad C(m,k)^{-1} \quad = \quad C\left(\tfrac{1}{m}\,,\,\tfrac{1}{k}\right)\,, \qquad\qquad$ *provided* $m,k\neq 0\,,$

(h) $\qquad [C(m,k)\,\mathbf{M}]:\mathbf{N} \quad = \quad \mathbf{M}:[C(m,k)\,\mathbf{N}]\,,$

(i) $\qquad \big[C(m,k)\,\mathbf{M}\big]:\mathbf{M} \quad \geq \quad \min\{m,k\}\;\mathbf{M}:\mathbf{M}\,,$

(j) $\qquad \big[C(m,k)\,\mathbf{M}\big]:\mathbf{M} \quad \leq \quad \max\{m,k\}\;\mathbf{M}:\mathbf{M}\,.$

(k) $\qquad C(m,k)$ *is diagonalizable. Its spectrum is* $\{m,k\}$*. The one-dimensional eigenspace corresponding to the eigenvalue* k *is spanned by* $\mathbf{I}$*. The eigenspace corresponding to the eigenvalue* m *is given by* $\{\mathbf{I}\}^\perp$ *with respect to the Frobenius inner product on* $\mathbb{R}^{3\times3}$ *or* $\mathbb{R}^{3\times3}_{\mathrm{sym}}$*. This is the set of all matrices* $\mathbf{M}$ *with* $\mathrm{trace}\,(\mathbf{M})=0$*.*

(l)

$$\max_{\mathbf{M}\in\mathbb{R}^{3\times3}_{\mathrm{sym}},\,\mathbf{M}\neq0} \frac{\sqrt{[C(m,k)\mathbf{M}]:[C(m,k)\mathbf{M}]}}{\sqrt{\mathbf{M}:\mathbf{M}}} \quad = \quad \max\big\{|m|,|k|\big\}$$

and analogously for $\mathbf{M}\in\mathbb{R}^{3\times3}$*.*

Proof. Let $m,k\in\mathbb{R}$, $\mathbf{M}_1,\mathbf{M}_2\in\mathbb{R}^{3\times3}$ and $\lambda\in\mathbb{R}$. Then

$$\begin{aligned}
&C(m,k)(\mathbf{M}_1+\lambda\mathbf{M}_2)\\
&\qquad = \quad m(\mathbf{M}_1+\lambda\mathbf{M}_2)+\frac{k-m}{3}\,\mathrm{trace}\,(\mathbf{M}_1+\lambda\mathbf{M}_2)\mathbf{I}\\
&\qquad = \quad m\mathbf{M}_1+\frac{k-m}{3}\,\mathrm{trace}\,(\mathbf{M}_1)\mathbf{I}+\lambda\Big(m\mathbf{M}_2+\frac{k-m}{3}\,\mathrm{trace}\,(\mathbf{M}_2)\mathbf{I}\Big)\\
&\qquad = \quad C(m,k)\mathbf{M}_1+\lambda C(m,k)\mathbf{M}_2\,.
\end{aligned}$$

So $C(m,k)\in\mathcal{L}(\mathbb{R}^{3\times3})$. If $\mathbf{M}\in\mathbb{R}^{3\times3}_{\mathrm{sym}}$ we also have $C(m,k)\mathbf{M}\in\mathbb{R}^{3\times3}_{\mathrm{sym}}$, so it even holds $C(m,k)\in\mathcal{L}(\mathbb{R}^{3\times3}_{\mathrm{sym}})$.

In the sequel let $m,\,m_1,\,m_2,\,k,\,k_1,\,k_2,\,\lambda\in\mathbb{R}$ and $\mathbf{M},\mathbf{N}\in\mathbb{R}^{3\times3}$.

(a)

$$
\begin{aligned}
&\big(C(m_1,k_1)+C(m_2,k_2)\big)\ \mathbf{M} \\
&\qquad = \quad C(m_1,k_1)\mathbf{M}+C(m_2,k_2)\mathbf{M} \\
&\qquad = \quad m_1\mathbf{M}+\frac{k_1-m_1}{3}\,\mathrm{trace}\,(\mathbf{M})\mathbf{I} \\
&\qquad\qquad +\ m_2\mathbf{M}+\frac{k_2-m_2}{3}\,\mathrm{trace}\,(\mathbf{M})\mathbf{I} \\
&\qquad = \quad (m_1+m_2)\mathbf{M}+\frac{(k_1+k_2)-(m_1+m_2)}{3}\,\mathrm{trace}\,(\mathbf{M})\mathbf{I} \\
&\qquad = \quad C(m_1+m_2,\ k_1+k_2)\ \mathbf{M}\,.
\end{aligned}
$$

(b)

$$
C(0,0)\,\mathbf{M} \quad = \quad 0\,\mathbf{M}+\frac{0-0}{3}\,\mathrm{trace}\,(\mathbf{M})\mathbf{I} \quad = \quad \mathbf{0}\,.
$$

(c)

$$
\begin{aligned}
&C(m_1,k_1)C(m_2,k_2)\,\mathbf{M} \\
&\qquad = \quad C(m_1,k_1)\left(m_2\mathbf{M}+\frac{k_2-m_2}{3}\,\mathrm{trace}\,(\mathbf{M})\mathbf{I}\right) \\
&\qquad = \quad m_1\left(m_2\mathbf{M}+\frac{k_2-m_2}{3}\,\mathrm{trace}\,(\mathbf{M})\mathbf{I}\right) \\
&\qquad\qquad +\ \frac{k_1-m_1}{3}\,\mathrm{trace}\left(m_2\mathbf{M}+\frac{k_2-m_2}{3}\,\mathrm{trace}\,(\mathbf{M})\mathbf{I}\right)\mathbf{I} \\
&\qquad = \quad m_1m_2\mathbf{M}+\frac{m_1k_2-m_1m_2}{3}\,\mathrm{trace}\,(\mathbf{M})\mathbf{I} \\
&\qquad\qquad +\ \frac{k_1-m_1}{3}\Big(m_2\,\mathrm{trace}\,(\mathbf{M})+(k_2-m_2)\,\mathrm{trace}\,(\mathbf{M})\Big)\,\mathbf{I} \\
&\qquad = \quad m_1m_2\mathbf{M}+\frac{k_1k_2-m_1m_2}{3}\,\mathrm{trace}\,(\mathbf{M})\mathbf{I} \\
&\qquad = \quad C(m_1m_2\,,\ k_1k_2)\,\mathbf{M}\,.
\end{aligned}
$$

(d)

$$
C(1,1)\,\mathbf{M} \quad = \quad 1\mathbf{M}+\frac{1-1}{3}\,\mathrm{trace}\,(\mathbf{M})\mathbf{I} \quad = \quad \mathbf{M}\,.
$$

(e)

$$
\begin{aligned}
\lambda C(m,k)\,\mathbf{M} \quad &= \quad \lambda\left(m\mathbf{M}+\frac{k-m}{3}\,\mathrm{trace}\,(\mathbf{M})\mathbf{I}\right) \\
&= \quad \lambda m\mathbf{M}\ +\ \frac{\lambda k-\lambda m}{3}\,\mathrm{trace}\,(\mathbf{M})\mathbf{I} \\
&= \quad C(\lambda m\,,\ \lambda k)\,\mathbf{M}\,.
\end{aligned}
$$

(f) According to part (c) ,

$$C(m_1, k_1)C(m_2, k_2)\,\mathbf{M} \quad = \quad C(m_1 m_2\,,\, k_1 k_2)\,\mathbf{M}\,,$$

which is symmetric in (m_1, k_1) and (m_2, k_2).

(g) Let $m, k \neq 0$. We use (c) and (d). Because of (f) it suffices to calculate

$$C\Big(\frac{1}{m}\,,\,\frac{1}{k}\Big)\,C(m,k) \quad = \quad C\Big(\frac{1}{m}m\,,\,\frac{1}{k}k\Big) \quad = \quad C(1,1) \quad = \quad \mathrm{Id}_{\mathcal{L}(\mathbb{R}^{3\times 3})}\,.$$

(h) Using the bilinearity of the Frobenius scalar product $\mathbf{M} : \mathbf{N}$ and also $\mathbf{I} : \mathbf{N} = \mathrm{trace}\,(\mathbf{N})$, we get

$$\begin{aligned}
[C(m,k)\mathbf{M}] : \mathbf{N} &= \Big(m\mathbf{M} + \frac{k-m}{3}\,\mathrm{trace}\,(\mathbf{M})\mathbf{I}\Big) : \mathbf{N} \\
&= m\mathbf{M} : \mathbf{N} + \frac{k-m}{3}\,\mathrm{trace}\,(\mathbf{M})\,\mathrm{trace}\,(\mathbf{N})\,.
\end{aligned}$$

As the Frobenius scalar product is symmetric the last term is symmetric in $\mathbf{M}$ and $\mathbf{N}$ and if we interchange $\mathbf{M}$ and $\mathbf{N}$ the first term is equal to $\mathbf{M} : \big[C(m,k)\,\mathbf{N}\big]$. This proves the statement.

(i) and (j)

$$\begin{aligned}
\big[C(m,k)\,\mathbf{M}\big] : \mathbf{M} &= \Big(m\mathbf{M} + \frac{k-m}{3}\,\mathrm{trace}\,(\mathbf{M})\mathbf{I}\Big) : \mathbf{M} \\
&= m\mathbf{M} : \mathbf{M} + \frac{k-m}{3}\,\mathrm{trace}\,(\mathbf{M})^2 \\
&= m\Big(\mathbf{M} : \mathbf{M} - \frac{1}{3}\,\mathrm{trace}\,(\mathbf{M})^2\Big) + \frac{k}{3}\,\mathrm{trace}\,(\mathbf{M})^2\,.
\end{aligned}$$

We will show that $\mathbf{M} : \mathbf{M} - \frac{1}{3}\,\mathrm{trace}\,(\mathbf{M})^2 \geq 0$. Then $\big[C(m,k)\,\mathbf{M}\big] : \mathbf{M}$ is nondecreasing in m for fixed k and nondecreasing in k for fixed m. So

$$\begin{aligned}
\big[C(m,k)\,\mathbf{M}\big] : \mathbf{M} &\geq \min\{m,k\}\Big(\mathbf{M} : \mathbf{M} - \frac{1}{3}\,\mathrm{trace}\,(\mathbf{M})^2\Big) \\
&\qquad + \frac{\min\{m,k\}}{3}\,\mathrm{trace}\,(\mathbf{M})^2 \\
&= \min\{m,k\}\;\mathbf{M} : \mathbf{M}\,,
\end{aligned}$$

which proves (i) and

$$\begin{aligned}
\big[C(m,k)\,\mathbf{M}\big] : \mathbf{M} &\leq \max\{m,k\}\Big(\mathbf{M} : \mathbf{M} - \frac{1}{3}\,\mathrm{trace}\,(\mathbf{M})^2\Big) \\
&\qquad + \frac{\max\{m,k\}}{3}\,\mathrm{trace}\,(\mathbf{M})^2 \\
&= \max\{m,k\}\;\mathbf{M} : \mathbf{M}\,,
\end{aligned}$$

which proves (j).

We are left with the proof of $\mathbf{M} : \mathbf{M} - \frac{1}{3}\,\mathrm{trace}\,(\mathbf{M})^2 \geq 0$. The Cauchy-Schwarz inequality yields

$$\begin{aligned}
\mathrm{trace}\,(\mathbf{M})^2 &= \left(\sum_{i=1}^{3} \mathbf{M}_{ii}\right)^2 = \left(\sum_{i=1}^{3} 1 \cdot \mathbf{M}_{ii}\right)^2 \\
&\leq \left(\sum_{i=1}^{3} 1^2\right)\left(\sum_{i=1}^{3} \mathbf{M}_{ii}^2\right) = 3 \sum_{i=1}^{3} \mathbf{M}_{ii}^2 \,.
\end{aligned}$$

Therefore

$$\mathbf{M} : \mathbf{M} - \frac{1}{3}\,\mathrm{trace}\,(\mathbf{M})^2 \geq \mathbf{M} : \mathbf{M} - \sum_{i=1}^{3} \mathbf{M}_{ii}^2 = \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \mathbf{M}_{ij}^2 \geq 0\,.$$

(k)

$$C(m,k)\,\mathbf{I} = m\,\mathbf{I} + \frac{k-m}{3}\,\mathrm{trace}\,(\mathbf{I})\,\mathbf{I} = m\,\mathbf{I} + (k-m)\,\mathbf{I} = k\,\mathbf{I}\,.$$

As $\mathbf{M} : \mathbf{I} = \mathrm{trace}\,(\mathbf{M})$ it is $\mathbf{M} \in \{\mathbf{I}\}^{\perp}$ equivalent to $\mathrm{trace}\,(\mathbf{M}) = 0$. And for $\mathbf{M} \in \mathbb{R}^{3\times 3}$ with $\mathrm{trace}\,(\mathbf{M}) = 0$ it holds

$$C(m,k)\mathbf{M} = m\mathbf{M} + \frac{k-m}{3}\,\mathrm{trace}\,(\mathbf{M})\mathbf{M} = m\mathbf{M}\,.$$

(l) It is $C(m,k)\mathbf{I} = k\mathbf{I}$. And for an eigenvector $\mathbf{M} \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ corresponding to the eigenvalue m it is $C(m,k)\mathbf{M} = m\mathbf{M}$. So

$$\begin{aligned}
&\max_{\mathbf{N} \in \mathbb{R}^{3\times 3}_{\mathrm{sym}},\, \mathbf{N} \neq 0} \frac{\sqrt{[C(m,k)\mathbf{N}] : [C(m,k)\mathbf{N}]}}{\sqrt{\mathbf{N} : \mathbf{N}}} \\
&\qquad \geq \max_{\mathbf{N} \in \{\mathbf{I},\mathbf{M}\}} \frac{\sqrt{[C(m,k)\mathbf{N}] : [C(m,k)\mathbf{N}]}}{\sqrt{\mathbf{N} : \mathbf{N}}} \\
&\qquad = \max\big\{|m|, |k|\big\}\,.
\end{aligned}$$

And from part (h), (c) and (j) it follows

$$\begin{aligned}
\big[C(m,k)\mathbf{N}\big] : \big[C(m,k)\mathbf{N}\big] &= \big[C(m^2,k^2)\mathbf{N}\big] : \mathbf{N} \\
&\leq \max\{m^2,\, k^2\}\, \mathbf{N} : \mathbf{N} \\
&= \max\big\{|m|, |k|\big\}^2\, \mathbf{N} : \mathbf{N}\,,
\end{aligned}$$

$\mathbf{N} \in \mathbb{R}^{3\times 3}$. Therefore also

$$\max_{\mathbf{N} \in \mathbb{R}^{3\times 3},\, \mathbf{N} \neq 0} \frac{\sqrt{[C(m,k)\mathbf{N}] : [C(m,k)\mathbf{N}]}}{\sqrt{\mathbf{N} : \mathbf{N}}} \leq \max\big\{|m|, |k|\big\}\,.$$

Finally, that ι_1, ι_2 from (5.15) are algebra homomorphisms and that the respective image $\{C(m,k) : m,k \in \mathbb{R}\}$ of these maps forms a commutative subalgebra with unity of $\mathcal{L}(\mathbb{R}^{3\times 3})$ and $\mathcal{L}(\mathbb{R}^{3\times 3}_{\mathrm{sym}})$, respectively, follows from (a), (c), (d) and (e). Furthermore, with (l) the statement $\iota_i(m,k) = 0$ implies $(m,k) = 0$ for $i = 1,2$. So ι_1, ι_2 actually are injective. □

Lemma 51. *For $\alpha, \beta \in L^\infty(D)$ the linear map*

$$\begin{aligned} \tilde{C}(\alpha,\beta) : \quad L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}}) \quad &\rightarrow \quad L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}}) \,, \\ \big[\tilde{C}(\alpha,\beta)\,\boldsymbol{\psi}\big](x) \quad &:= \quad C\big(\alpha(x),\beta(x)\big)\,\boldsymbol{\psi}(x) \,, \qquad\qquad x \in D \,, \end{aligned}$$

is well-defined and bounded with

$$\|\tilde{C}(\alpha,\beta)\|_{\mathcal{L}(L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}}))} \quad = \quad \max\big\{\|\alpha\|_{L^\infty(D)}\,,\,\|\beta\|_{L^\infty(D)}\big\}\,. \tag{5.16}$$

Proof. For any $\boldsymbol{\psi} \in L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}})$, Lemma 50 (h), (c) and (j) yield

$$\begin{aligned} \|\tilde{C}(\alpha,\beta)\,\boldsymbol{\psi}\|^2_{L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}})} \quad &= \quad \int_D \big[\tilde{C}(\alpha,\beta)\,\boldsymbol{\psi}\big] : \big[\tilde{C}(\alpha,\beta)\,\boldsymbol{\psi}\big]\,dx \\ &= \quad \int_D \big[\tilde{C}(\alpha,\beta)^2\,\boldsymbol{\psi}\big] : \boldsymbol{\psi}\,dx \\ &= \quad \int_D \big[\tilde{C}\big(\alpha^2,\beta^2\big)\,\boldsymbol{\psi}\big] : \boldsymbol{\psi}\,dx \\ &\leq \quad \int_D \max\big\{\alpha^2\,,\,\beta^2\big\}\,\boldsymbol{\psi} : \boldsymbol{\psi}\,dx \\ &\leq \quad \max\big\{\|\alpha\|_{L^\infty(D)}\,,\,\|\beta\|_{L^\infty(D)}\big\}^2 \int_D \boldsymbol{\psi} : \boldsymbol{\psi}\,dx\,. \end{aligned}$$

Thus

$$\|\tilde{C}(\alpha,\beta)\,\boldsymbol{\psi}\|_{L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}})} \quad \leq \quad \max\big\{\|\alpha\|_{L^\infty(D)}\,,\,\|\beta\|_{L^\infty(D)}\big\}\,\|\boldsymbol{\psi}\|_{L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}})}$$

and therefore

$$\|\tilde{C}(\alpha,\beta)\|_{\mathcal{L}(L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}}))} \quad \leq \quad \max\big\{\|\alpha\|_{L^\infty(D)}\,,\,\|\beta\|_{L^\infty(D)}\big\}\,.$$

To show that even equality holds, we proceed as follows.

In the case: $\max\big\{\|\alpha\|_{L^\infty(D)}\,,\,\|\beta\|_{L^\infty(D)}\big\} = \|\alpha\|_{L^\infty(D)}$ we choose a constant function $\psi : D \rightarrow \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ such that for every $x \in D$ the matrix $\psi(x)$ is an eigenvector of $C\big(\alpha(x),\beta(x)\big)$ corresponding to the eigenvalue $\alpha(x)$. According to Lemma 50(k) that means $\psi(x) \in \{\mathbf{I}\}^\perp$. A possible choice for $\psi(x) =: (\psi(x)_{ij})_{i,j=1,2,3}$ would be: $\psi(x)_{13} = \psi(x)_{31} = 1$ and $\psi(x)_{ij} = 0$ for $(i,j) \notin \{(1,3),(3,1)\}$. Let $D_n := \{x \in D :$

$|\alpha(x)|^2 > \|\alpha\|^2_{L^\infty(D)} - \frac{1}{n}\} \cap B(\delta_n, 0),\ n \in \mathbb{N}$, where $B(\delta_n, 0)$ is a ball around 0 with a radius δ_n which is big enough for the Lebesgue measure $\lambda^3(D_n) > 0$. Let 1_{D_n} denote the characteristic function of D_n and $\phi_n := \frac{1}{\|1_{D_n}\psi\|_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})}} 1_{D_n}\psi,\ n \in \mathbb{N}$. Then $\|\phi_n\|_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})} = 1$ and

$$\begin{aligned}
\|\tilde{C}(\alpha,\beta)\,\phi_n\|^2_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})} &= \frac{\|\alpha 1_{D_n}\psi\|^2_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})}}{\|1_{D_n}\psi\|^2_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})}} \\
&\geq \Big(\|\alpha\|^2_{L^\infty(D)} - \frac{1}{n}\Big) \frac{\|1_{D_n}\psi\|^2_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})}}{\|1_{D_n}\psi\|^2_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})}} \\
&= \|\alpha\|^2_{L^\infty(D)} - \frac{1}{n}\,, \qquad n \in \mathbb{N}\,.
\end{aligned}$$

So

$$\sup_{n\in\mathbb{N}} \|\tilde{C}(\alpha,\beta)\,\phi_n\|_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})} \quad \geq \quad \|\alpha\|_{L^\infty(D)}$$

and therefore

$$\|\tilde{C}(\alpha,\beta)\|_{\mathcal{L}(L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}}))} \quad \geq \quad \|\alpha\|_{L^\infty(D)} \quad = \quad \max\big\{\|\alpha\|_{L^\infty(D)}\,,\, \|\beta\|_{L^\infty(D)}\big\}\,.$$

In the case: $\max\big\{\|\alpha\|_{L^\infty(D)}\,,\, \|\beta\|_{L^\infty(D)}\big\} = \|\beta\|_{L^\infty(D)}$ we repeat this calculation with ψ substituted by $x \mapsto \mathbf{I}$, since due to Lemma 50(k), for every $x \in D$ the matrix $\mathbf{I}$ is an eigenvector of $C\big(\alpha(x), \beta(x)\big)$ corresponding to the eigenvalue $\beta(x)$. $\square$

Notation 52. Henceforth we drop the ˜ in the notation of the maps $\tilde{C}(\alpha,\beta)$ defined in Lemma 51 and use the same notation for $C(m,k)$ with $m,k \in \mathbb{R}$ and $\tilde{C}(\alpha,\beta)$ with $\alpha,\beta \in L^\infty(D)$. It should be possible to distinguish the two kinds of mathematical objects by the surrounding context. $\square$

Lemma 53.

(a) The bounded linear maps $C(\mu_H, \kappa_H)$, $C(\mu_{M,l}\,,\, \kappa_{M,l}) \in \mathcal{L}\big(L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})\big)$ *are invertible with*

$$\begin{aligned}
C(\mu_H, \kappa_H)^{-1} &= C\Big(\frac{1}{\mu_H}\,,\, \frac{1}{\kappa_H}\Big)\,, \\
C(\mu_{M,l}\,,\, \kappa_{M,l})^{-1} &= C\Big(\frac{1}{\mu_{M,l}}\,,\, \frac{1}{\kappa_{M,l}}\Big)
\end{aligned}$$

for $l = 1, \ldots, L$.

(b) For $\boldsymbol{\psi} \in L^2(D, \mathbb{R}^{3\times3}_{\text{sym}})$ the following estimates hold:

$$\begin{aligned} \min\left\{\frac{1}{\mu_H}, \frac{1}{\kappa_H}\right\} \boldsymbol{\psi} : \boldsymbol{\psi} &\leq \left[C(\mu_H, \kappa_H)^{-1}\boldsymbol{\psi}\right] : \boldsymbol{\psi} \\ &\leq \max\left\{\frac{1}{\mu_H}, \frac{1}{\kappa_H}\right\} \boldsymbol{\psi} : \boldsymbol{\psi}\,, \end{aligned} \tag{5.17}$$

$$\begin{aligned} \min\left\{\frac{1}{\mu_{M,l}}, \frac{1}{\kappa_{M,l}}\right\} \boldsymbol{\psi} : \boldsymbol{\psi} &\leq \left[C(\mu_{M,l}\,,\, \kappa_{M,l})^{-1}\boldsymbol{\psi}\right] : \boldsymbol{\psi} \\ &\leq \max\left\{\frac{1}{\mu_{M,l}}, \frac{1}{\kappa_{M,l}}\right\} \boldsymbol{\psi} : \boldsymbol{\psi}\,, \end{aligned} \tag{5.18}$$

$l = 1, \ldots, L$, pointwise almost everywhere on D.

Proof. (a) As due to (5.13), μ_H, $\mu_{M,1}, \ldots, \mu_{M,L}$ and κ_H, $\kappa_{M,1}, \ldots, \kappa_{M,L}$ are greater than 0 almost everywhere, this follows from Lemma 50(g) with (m, k) substituted by $\big(\mu_H(x), \kappa_H(x)\big)$ and $\big(\mu_{M,l}(x),\, \kappa_{M,l}(x)\big)$, respectively, for any $x \in D$.

(b) This is the statement of Lemma 50(i) and (j) with (m, k) substituted by $\big(1/\mu_H(x), 1/\kappa_H(x)\big)$ and $\big(1/\mu_{M,l}(x),\, 1/\kappa_{M,l}(x)\big)$ respectively for $x \in D$.

□

Lemma 54. *The following expressions define scalar products on $L^2(D, \mathbb{R}^{3\times3}_{\text{sym}})$:*

$$\Big(C(\mu_H, \kappa_H)^{-1}\boldsymbol{\sigma}^{(1)}\,,\, \boldsymbol{\sigma}^{(2)}\Big)_{L^2(D,\mathbb{R}^{3\times3}_{\text{sym}})}\,, \qquad \Big(C(\mu_{M,l}\,,\, \kappa_{M,l})^{-1}\boldsymbol{\sigma}^{(1)}\,,\, \boldsymbol{\sigma}^{(2)}\Big)_{L^2(D,\mathbb{R}^{3\times3}_{\text{sym}})}$$

for all $\boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)} \in L^2(D, \mathbb{R}^{3\times3}_{\text{sym}})$ and $l = 1, \ldots, L$.

They are equivalent to $(\cdot\,,\cdot)_{L^2(D,\mathbb{R}^{3\times3}_{\text{sym}})}$, i.e.

$$q_H\,(\boldsymbol{\sigma}, \boldsymbol{\sigma})_{L^2(D,\mathbb{R}^{3\times3}_{\text{sym}})} \leq \big(C(\mu_H, \kappa_H)^{-1}\boldsymbol{\sigma}\,,\, \boldsymbol{\sigma}\big)_{L^2(D,\mathbb{R}^{3\times3}_{\text{sym}})} \leq Q_H\,(\boldsymbol{\sigma}, \boldsymbol{\sigma})_{L^2(D,\mathbb{R}^{3\times3}_{\text{sym}})}\,,$$

$\boldsymbol{\sigma} \in L^2(D, \mathbb{R}^{3\times3}_{\text{sym}})$ with

$$q_H = \min\left\{\frac{1}{\|\mu_H\|_{L^\infty(D)}}, \frac{1}{\|\kappa_H\|_{L^\infty(D)}}\right\}, \tag{5.19}$$

$$Q_H = \max\left\{\frac{1}{\mu_{H,0}}, \frac{1}{\kappa_{H,0}}\right\} \tag{5.20}$$

and

$$\begin{aligned} q_{M,l}\,(\boldsymbol{\sigma}, \boldsymbol{\sigma})_{L^2(D,\mathbb{R}^{3\times3}_{\text{sym}})} &\leq \big(C(\mu_{M,l}, \kappa_{M,l})^{-1}\boldsymbol{\sigma}\,,\, \boldsymbol{\sigma}\big)_{L^2(D,\mathbb{R}^{3\times3}_{\text{sym}})} \\ &\leq Q_{M,l}\,(\boldsymbol{\sigma}, \boldsymbol{\sigma})_{L^2(D,\mathbb{R}^{3\times3}_{\text{sym}})}\,, \end{aligned}$$

$\boldsymbol{\sigma} \in L^2(D, \mathbb{R}^{3\times3}_{\mathrm{sym}})$ *with*

$$q_{M,l} = \min\left\{\frac{1}{\|\mu_{M,l}\|_{L^\infty(D)}},\ \frac{1}{\|\kappa_{M,l}\|_{L^\infty(D)}}\right\}, \tag{5.21}$$

$$Q_{M,l} = \max\left\{\frac{1}{\mu_{M,l,0}},\ \frac{1}{\kappa_{M,l,0}}\right\}, \tag{5.22}$$

for $l = 1, \ldots, L$.

Proof. Bilinearity follows from the linearity of the respective maps $C(\mu_H, \kappa_H)^{-1}$ and $C(\mu_{M,l}, \kappa_{M,l})^{-1}$, $l = 1, \ldots, L$. Symmetry follows from Lemma 50(h), and the estimates concerning the equivalence of the scalar products hold because of our parameter restrictions (5.13) together with (5.17) and (5.18). This also proves the positive definiteness of the newly defined scalar products. □

Remark 55. *We recall that in* (2.11) *we rescaled the physical shear modulus* $\tilde{\mu}$ *and bulk modulus* $\tilde{\kappa}$ *as* $\mu = 2\tilde{\mu}$ *and* $\kappa = 3\tilde{\kappa}$. *Expressed in physical variables, the elastic stiffness tensor therefore reads* $C(\mu, \kappa) = C(2\tilde{\mu}, 3\tilde{\kappa})$.

Furthermore in section 2.3 we mentioned another material parameter: Lamé's First Parameter $\lambda = \tilde{\kappa} - (2/3)\tilde{\mu}$. *Equivalently it holds* $3\tilde{\kappa} = 3\lambda + 2\tilde{\mu}$. *Hence* $C(\mu, \kappa) = C(2\tilde{\mu},\, 3\lambda + 2\tilde{\mu})$.

Widely spread in literature is the definition of the elastic stiffness tensor as a function $C' : \{\, (\tilde{\mu}, \lambda) :\ \tilde{\mu} > 0,\ \lambda > -(2/3)\tilde{\mu}\,\} \to \mathcal{L}(\mathbb{R}^{3\times3}_{\mathrm{sym}})$ *depending on the two material parameters* $\tilde{\mu}$ *and* λ. *That is,*

$$\begin{aligned} C'(\tilde{\mu}, \lambda)\mathbf{M} &:= C(2\tilde{\mu},\, 3\lambda + 2\tilde{\mu})\mathbf{M} \\ &= 2\tilde{\mu}\mathbf{M} + \frac{(3\lambda + 2\tilde{\mu}) - 2\tilde{\mu}}{3}\,\mathrm{trace}\,(\mathbf{M})\mathbf{I} \\ &= 2\tilde{\mu}\mathbf{M} + \lambda\,\mathrm{trace}\,(\mathbf{M})\mathbf{I}\,, \end{aligned}$$

$\mathbf{M} \in \mathbb{R}^{3\times3}_{\mathrm{sym}}$, *where we used* (5.14).

In this parametrization, however, the concatenation of two such linear maps reads

$$\begin{aligned} C'(\tilde{\mu}_1, \lambda_1)C'(\tilde{\mu}_2, \lambda_2)\mathbf{M} &= C'(\tilde{\mu}_1, \lambda_1)\Big(2\tilde{\mu}_2\mathbf{M} + \lambda_2\,\mathrm{trace}\,(\mathbf{M})\mathbf{I}\Big) \\ &= 2(2\tilde{\mu}_1\tilde{\mu}_2)\mathbf{M} + (2\tilde{\mu}_1\lambda_2 + 2\tilde{\mu}_2\lambda_1 + 3\lambda_1\lambda_2)\,\mathrm{trace}\,(\mathbf{M})\mathbf{I} \\ &= C'\big(2\tilde{\mu}_1\tilde{\mu}_2\,,\ 2\tilde{\mu}_1\lambda_2 + 2\tilde{\mu}_2\lambda_1 + 3\lambda_1\lambda_2\big)\mathbf{M}\,. \end{aligned}$$

Also it is

$$\mathrm{Id}_{\mathcal{L}(\mathbb{R}^{3\times3}_{\mathrm{sym}})} = C(1,1) = C'\Big(\frac{1}{2}\,, 0\Big)\,.$$

Consequently

$$C'(\tilde{\mu},\lambda)^{-1} \quad = \quad C'\Big(\frac{1}{4\tilde{\mu}}\,,\,-\frac{\lambda}{2\tilde{\mu}(2\tilde{\mu}+3\lambda)}\Big)\,.$$

With a look at Lemma 50 (c), (d) *and* (g) *we see that in this parametrization calculations are much more tedious than with the use of definition* (5.14).[1] □

5.3 Transformation of Variables

To prove existence, uniqueness and stability of the solution $(\mathbf{v},\boldsymbol{\sigma},\boldsymbol{\eta})^\top$ of the initial-boundary value problem (5.1),

$$\begin{aligned}
\mathbf{v}'(t) \;&=\; \vartheta\,\mathrm{div}\,\boldsymbol{\sigma}(t) \;+\; \tilde{\mathbf{f}}(t)\,,\\
\boldsymbol{\sigma}'(t) \;&=\; C\Big(\mu_H+\sum_{l=1}^{L}\mu_{M,l}\,,\,\kappa_H+\sum_{l=1}^{L}\kappa_{M,l}\Big)\,\varepsilon\big(\mathbf{v}(t)\big) \;+\; \sum_{l=1}^{L}\boldsymbol{\eta}_l(t) \;+\; \mathbf{g}(t)\,,\\
\boldsymbol{\eta}_l'(t) \;&=\; -\omega_{\boldsymbol{\sigma},l}C\big(\mu_{M,l}\,,\,\kappa_{M,l}\big)\,\varepsilon\big(\mathbf{v}(t)\big) \;-\; \omega_{\boldsymbol{\sigma},l}\boldsymbol{\eta}_l(t)\,, \qquad l=1,\ldots,L\,,
\end{aligned} \tag{5.23}$$

$$\mathbf{v}(0) \;=\; \mathbf{v}^{(0)}\,,\qquad \boldsymbol{\sigma}(0) \;=\; \boldsymbol{\sigma}^{(0)}\,,\qquad \boldsymbol{\eta}(0) \;=\; \boldsymbol{\eta}^{(0)}\,,$$
$$\mathbf{v}(t)\big|_{\partial D_D} \;=\; \mathbf{0}\,,\qquad \mathbf{n}^\top\boldsymbol{\sigma}(t)\big|_{\partial D_N} \;=\; \mathbf{0}\,,$$

$t\in[0,t_1]$, which we interpret as the evolution equation (5.2) on the spaces X as in (5.4) and $\mathcal{D}(A)$ as in (5.11), we are going to apply Theorem 3 of section 3.1.

To find a suitable scalar product $(.\,,.)_T$, for which the operator A in (5.3) is monotone in the sense of (3.3), that is $(Au,u)_T\geq 0$, $u\in\mathcal{D}(A)$, we apply a variable transformation T to (5.23) in a first step in this section. This transformation is a concrete instance of the abstract transformation T in section 4.1. As we will see, the new variables will have a physical meaning and also the mechanical energy of the system and its decline over time can be clearly specified.

Definition 56. *The linear transformation $T: X\to X$ is defined as*

$$T\begin{pmatrix}\mathbf{v}\\ \boldsymbol{\sigma}\\ \boldsymbol{\eta}_1\\ \vdots\\ \boldsymbol{\eta}_L\end{pmatrix} \quad = \quad \begin{pmatrix}\mathbf{v}\\ \boldsymbol{\sigma}+\sum_{l=1}^{L}\frac{1}{\omega_{\boldsymbol{\sigma},l}}\boldsymbol{\eta}_l\\ -\frac{1}{\omega_{\boldsymbol{\sigma},1}}\boldsymbol{\eta}_1\\ \vdots\\ -\frac{1}{\omega_{\boldsymbol{\sigma},L}}\boldsymbol{\eta}_L\end{pmatrix}\,. \tag{5.24}$$

□

[1] Many thanks to Johann Bitzenbauer. As an expert in mechanics he told me about $\tilde{\mu}$ and $\tilde{\kappa}$ being the physically natural pair of material parameters. It turned out to be a good choice.

Lemma 57. *The linear operator $T : \big(X, \|.\|_X\big) \to \big(X, \|.\|_X\big)$ defined in (5.24) is bounded and boundedly invertible with T^{-1} given by*

$$T^{-1}\begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma}_H \\ \boldsymbol{\sigma}_{M,1} \\ \vdots \\ \boldsymbol{\sigma}_{M,L} \end{pmatrix} = \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma}_H + \sum_{l=1}^L \boldsymbol{\sigma}_{M,l} \\ -\omega_{\boldsymbol{\sigma},1}\boldsymbol{\sigma}_{M,1} \\ \vdots \\ -\omega_{\boldsymbol{\sigma},L}\boldsymbol{\sigma}_{M,L} \end{pmatrix}. \tag{5.25}$$

Proof. First we prove that T is invertible. For $u = (\mathbf{v}, \boldsymbol{\sigma}, \boldsymbol{\eta})^\top \in X$ we have

$$\begin{aligned} T^{-1}T\,u &= T^{-1}\begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma} + \sum_{l=1}^L \frac{1}{\omega_{\boldsymbol{\sigma},l}}\boldsymbol{\eta}_l \\ -\frac{1}{\omega_{\boldsymbol{\sigma},1}}\boldsymbol{\eta}_1 \\ \vdots \\ -\frac{1}{\omega_{\boldsymbol{\sigma},L}}\boldsymbol{\eta}_L \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{v} \\ \Big(\boldsymbol{\sigma} + \sum_{l=1}^L \frac{1}{\omega_{\boldsymbol{\sigma},l}}\boldsymbol{\eta}_l\Big) + \sum_{l=1}^L \big(-\frac{1}{\omega_{\boldsymbol{\sigma},l}}\boldsymbol{\eta}_l\big) \\ -\omega_{\boldsymbol{\sigma},1}\big(-\frac{1}{\omega_{\boldsymbol{\sigma},1}}\boldsymbol{\eta}_1\big) \\ \vdots \\ -\omega_{\boldsymbol{\sigma},L}\big(-\frac{1}{\omega_{\boldsymbol{\sigma},L}}\boldsymbol{\eta}_L\big) \end{pmatrix} \\ &= \big(\mathbf{v},\, \boldsymbol{\sigma},\, \boldsymbol{\eta}_1,\, \ldots,\, \boldsymbol{\eta}_L\big)^\top \\ &= u\,. \end{aligned}$$

And for $w = (\mathbf{v}, \boldsymbol{\sigma}_H, \boldsymbol{\sigma}_M)^\top \in X$ it holds

$$\begin{aligned} TT^{-1}\,w &= T\begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma}_H + \sum_{l=1}^L \boldsymbol{\sigma}_{M,l} \\ -\omega_{\boldsymbol{\sigma},1}\boldsymbol{\sigma}_{M,1} \\ \vdots \\ -\omega_{\boldsymbol{\sigma},L}\boldsymbol{\sigma}_{M,L} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{v} \\ \Big(\boldsymbol{\sigma}_H + \sum_{l=1}^L \boldsymbol{\sigma}_{M,l}\Big) + \sum_{l=1}^L \frac{1}{\omega_{\boldsymbol{\sigma},l}}\big(-\omega_{\boldsymbol{\sigma},l}\boldsymbol{\sigma}_{M,l}\big) \\ -\frac{1}{\omega_{\boldsymbol{\sigma},1}}\big(-\omega_{\boldsymbol{\sigma},1}\boldsymbol{\sigma}_{M,1}\big) \\ \vdots \\ -\frac{1}{\omega_{\boldsymbol{\sigma},L}}\big(-\omega_{\boldsymbol{\sigma},L}\boldsymbol{\sigma}_{M,L}\big) \end{pmatrix} \\ &= \big(\mathbf{v},\, \boldsymbol{\sigma}_H,\, \boldsymbol{\sigma}_{M,1},\, \ldots,\, \boldsymbol{\sigma}_{M,L}\big)^\top \end{aligned}$$

$$= w\,.$$

Next we prove that T is bounded. For $u = (\mathbf{v}, \boldsymbol{\sigma}, \boldsymbol{\eta})^\top \in X$ we have

$$\begin{aligned}
&\|Tu\|_X^2\\
&= \Big\| \Big(\mathbf{v}\,,\, \boldsymbol{\sigma} + \sum_{l=1}^{L} \frac{1}{\omega_{\boldsymbol{\sigma},l}} \boldsymbol{\eta}_l\,,\, -\frac{1}{\omega_{\boldsymbol{\sigma},1}} \boldsymbol{\eta}_1\,,\, \dots\,,\, -\frac{1}{\omega_{\boldsymbol{\sigma},L}} \boldsymbol{\eta}_L \Big)^\top \Big\|_X^2\\
&= \|\mathbf{v}\|^2_{L^2(D,\mathbb{R}^3)} + \Big\| \boldsymbol{\sigma} + \sum_{l=1}^{L} \frac{1}{\omega_{\boldsymbol{\sigma},l}} \boldsymbol{\eta}_l \Big\|^2_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})} + \sum_{l=1}^{L} \| -\frac{1}{\omega_{\boldsymbol{\sigma},l}} \boldsymbol{\eta}_l \|^2_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})}\\
&\leq \|\mathbf{v}\|^2_{L^2(D,\mathbb{R}^3)} + (L+1) \Big(\|\boldsymbol{\sigma}\|^2_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})} + \sum_{l=1}^{L} \Big\| \frac{1}{\omega_{\boldsymbol{\sigma},l}} \boldsymbol{\eta}_l \Big\|^2_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})} \Big)\\
&\quad + \sum_{l=1}^{L} \| -\frac{1}{\omega_{\boldsymbol{\sigma},l}} \boldsymbol{\eta}_l \|^2_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})}\\
&\leq \|\mathbf{v}\|^2_{L^2(D,\mathbb{R}^3)} + (L+1) \|\boldsymbol{\sigma}\|^2_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})} + (L+2) \sum_{l=1}^{L} \frac{1}{\omega^2_{\boldsymbol{\sigma},l,0}} \|\boldsymbol{\eta}_l\|^2_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})}\\
&\leq \max\Big\{ L+1,\, \frac{L+2}{\omega^2_{\boldsymbol{\sigma},1,0}},\, \dots,\, \frac{L+2}{\omega^2_{\boldsymbol{\sigma},L,0}} \Big\}\, \|u\|_X^2\,.
\end{aligned}$$

Here the first estimate has been done by applying the Cauchy-Schwarz inequality to the second summand.

Finally, the boundedness of T^{-1} follows from the open mapping theorem. Alternatively it can be proven in an elementary way as follows. For $w = (\mathbf{v}, \boldsymbol{\sigma}_H, \boldsymbol{\sigma}_M)^\top \in X$ we get

$$\begin{aligned}
&\|T^{-1}w\|_X^2\\
&= \Big\| \Big(\mathbf{v}\,,\, \boldsymbol{\sigma}_H + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l}\,,\, -\omega_{\boldsymbol{\sigma},1} \boldsymbol{\sigma}_{M,1}\,,\, \dots\,,\, -\omega_{\boldsymbol{\sigma},L} \boldsymbol{\sigma}_{M,L} \Big)^\top \Big\|_X^2\\
&= \|\mathbf{v}\|^2_{L^2(D,\mathbb{R}^3)} + \Big\| \boldsymbol{\sigma}_H + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l} \Big\|^2_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})} + \sum_{l=1}^{L} \| -\omega_{\boldsymbol{\sigma},l} \boldsymbol{\sigma}_{M,l} \|^2_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})}\\
&\leq \|\mathbf{v}\|^2_{L^2(D,\mathbb{R}^3)} + (L+1) \Big(\|\boldsymbol{\sigma}_H\|^2_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})} + \sum_{l=1}^{L} \|\boldsymbol{\sigma}_{M,l}\|^2_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})} \Big)\\
&\quad + \sum_{l=1}^{L} \|\omega_{\boldsymbol{\sigma},l}\|^2_{L^\infty(D)} \|\boldsymbol{\sigma}_{M,l}\|^2_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})}\\
&\leq \|\mathbf{v}\|^2_{L^2(D,\mathbb{R}^3)} + (L+1) \|\boldsymbol{\sigma}_H\|^2_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})}
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{l=1}^{L} \big(L+1+\|\omega_{\boldsymbol{\sigma},l}\|^2_{L^\infty(D)}\big)\|\boldsymbol{\sigma}_{M,l}\|^2_{L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}})} \\
\leq \quad & \Big(L+1+\max\big\{\|\omega_{\boldsymbol{\sigma},1}\|^2_{L^\infty(D)},\ \ldots,\ \|\omega_{\boldsymbol{\sigma},L}\|^2_{L^\infty(D)}\big\}\Big)\ \|w\|^2_X\,.
\end{aligned}
$$

□

Proposition 58. *With A, $\mathcal{D}(A)$ and T from* (5.3), (5.11) *and* (5.24), *respectively, we define*

$$B \ \coloneqq\ TAT^{-1} \tag{5.26}$$

and

$$\mathcal{D}(B) \ \coloneqq\ T\mathcal{D}(A)\,. \tag{5.27}$$

in analogy to the abstract definitions in section 4.1. Then

$$
\begin{aligned}
\mathcal{D}(B) \quad = \quad & \Big\{(\mathbf{v},\boldsymbol{\sigma}_H,\boldsymbol{\sigma}_M)^\top \in\ V\times L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})\times L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})^L \ : \\
& \qquad\qquad \boldsymbol{\sigma}_H+\sum_{l=1}^{L}\boldsymbol{\sigma}_{M,l}\in S\Big\}
\end{aligned} \tag{5.28}
$$

with V from (5.9) *and S from* (5.10) *and*

$$
B\begin{pmatrix}\mathbf{v}\\ \boldsymbol{\sigma}_H\\ \boldsymbol{\sigma}_{M,1}\\ \vdots\\ \boldsymbol{\sigma}_{M,L}\end{pmatrix} \quad = \quad -\begin{pmatrix}\vartheta\,\mathrm{div}\Big(\boldsymbol{\sigma}_H+\sum_{l=1}^{L}\boldsymbol{\sigma}_{M,l}\Big)\\ C\big(\mu_H,\kappa_H\big)\,\varepsilon\big(\mathbf{v}\big)\\ C\big(\mu_{M,1},\kappa_{M,1}\big)\,\varepsilon(\mathbf{v})-\omega_{\boldsymbol{\sigma},1}\boldsymbol{\sigma}_{M,1}\\ \vdots\\ C\big(\mu_{M,L},\kappa_{M,L}\big)\,\varepsilon(\mathbf{v})-\omega_{\boldsymbol{\sigma},L}\boldsymbol{\sigma}_{M,L}\end{pmatrix}, \tag{5.29}
$$

$(\mathbf{v},\boldsymbol{\sigma}_H,\boldsymbol{\sigma}_M)^\top\in\mathcal{D}(B)$. *Here and in the sequel we use the abbreviation*

$$\boldsymbol{\sigma}_M \ \coloneqq\ \big(\boldsymbol{\sigma}_{M,1},\ldots,\boldsymbol{\sigma}_{M,L}\big)^\top.$$

Proof. First we prove (5.28).

"$\subseteq$": For $u=(\mathbf{v},\boldsymbol{\sigma},\boldsymbol{\eta})^\top\in\mathcal{D}(A)=V\times S\times L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})^L$,

$$
\begin{pmatrix}\mathbf{v}\\ \boldsymbol{\sigma}_H\\ \boldsymbol{\sigma}_{M,1}\\ \vdots\\ \boldsymbol{\sigma}_{M,L}\end{pmatrix} \ \coloneqq\ Tu \ =\ \begin{pmatrix}\mathbf{v}\\ \boldsymbol{\sigma}+\sum_{l=1}^{L}\frac{1}{\omega_{\boldsymbol{\sigma},l}}\boldsymbol{\eta}_l\\ -\frac{1}{\omega_{\boldsymbol{\sigma},1}}\boldsymbol{\eta}_1\\ \vdots\\ -\frac{1}{\omega_{\boldsymbol{\sigma},L}}\boldsymbol{\eta}_L\end{pmatrix} \ \in\ V\times L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})\times L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})^L
$$

and $\boldsymbol{\sigma}_H + \sum_{l=1}^L \boldsymbol{\sigma}_{M,l} = \boldsymbol{\sigma} \in S$, so Tu is an element of the set on the right-hand side of (5.28).

"$\supseteq$": Vice versa, for $w = (\mathbf{v}, \boldsymbol{\sigma}_H, \boldsymbol{\sigma}_M)^\top$ being an element of the set on the right-hand side of (5.28) it holds

$$T^{-1}w \quad = \quad \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma}_H + \sum_{l=1}^L \boldsymbol{\sigma}_{M,l} \\ -\omega_{\boldsymbol{\sigma},1}\boldsymbol{\sigma}_{M,1} \\ \vdots \\ -\omega_{\boldsymbol{\sigma},L}\boldsymbol{\sigma}_{M,L} \end{pmatrix} \quad \in \quad V \times S \times L^2(D, \mathbb{R}^{3\times 3}_{\mathrm{sym}})^L \quad = \quad \mathcal{D}(A)\,.$$

So $w = TT^{-1}w \in \mathcal{D}(B)$.

Now we show (5.29). With $w = (\mathbf{v}, \boldsymbol{\sigma}_H, \boldsymbol{\sigma}_M)^\top \in \mathcal{D}(B)$ and the formula for A in (5.3), for T^{-1} in (5.25) and for T in (5.24) we simply calculate

$$\begin{aligned} & TAT^{-1}\,w \\ = \quad & TA \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma}_H + \sum_{l=1}^L \boldsymbol{\sigma}_{M,l} \\ -\omega_{\boldsymbol{\sigma},1}\boldsymbol{\sigma}_{M,1} \\ \vdots \\ -\omega_{\boldsymbol{\sigma},L}\boldsymbol{\sigma}_{M,L} \end{pmatrix} \\ = \quad & -T \begin{pmatrix} \vartheta \operatorname{div}\Big(\boldsymbol{\sigma}_H + \sum_{l=1}^L \boldsymbol{\sigma}_{M,l}\Big) \\ C\Big(\mu_H + \sum_{l=1}^L \mu_{M,l}\ ,\ \kappa_H + \sum_{l=1}^L \kappa_{M,l}\Big)\,\varepsilon(\mathbf{v}) \ - \ \sum_{l=1}^L \omega_{\boldsymbol{\sigma},l}\boldsymbol{\sigma}_{M,l} \\ -\omega_{\boldsymbol{\sigma},1} C\big(\mu_{M,1}\ ,\ \kappa_{M,1}\big)\,\varepsilon(\mathbf{v}) \ + \ \omega_{\boldsymbol{\sigma},1}^2\boldsymbol{\sigma}_{M,1} \\ \vdots \\ -\omega_{\boldsymbol{\sigma},L} C\big(\mu_{M,L}\ ,\ \kappa_{M,L}\big)\,\varepsilon(\mathbf{v}) \ + \ \omega_{\boldsymbol{\sigma},L}^2\boldsymbol{\sigma}_{M,L} \end{pmatrix} \\ = \quad & - \begin{pmatrix} \vartheta \operatorname{div}\Big(\boldsymbol{\sigma}_H + \sum_{l=1}^L \boldsymbol{\sigma}_{M,l}\Big) \\ C\big(\mu_H, \kappa_H\big)\,\varepsilon(\mathbf{v}) \\ C\big(\mu_{M,1}, \kappa_{M,1}\big)\,\varepsilon(\mathbf{v}) - \omega_{\boldsymbol{\sigma},1}\boldsymbol{\sigma}_{M,1} \\ \vdots \\ C\big(\mu_{M,L}, \kappa_{M,L}\big)\,\varepsilon(\mathbf{v}) - \omega_{\boldsymbol{\sigma},L}\boldsymbol{\sigma}_{M,L} \end{pmatrix}, \end{aligned}$$

which is the expression on the right-hand side of (5.29). $\square$

Lemma 59. *The linear operator $B : X \supseteq \mathcal{D}(B) \to X$ defined in* (5.26) *can be decomposed into*

$$B \ = \ -P_1 Q + P_2\,,$$

where $P_1, P_2 : X \to X$,

$$P_1 w \;:=\; \begin{pmatrix} \vartheta\,\mathbf{v} \\ C(\mu_H, \kappa_H)\,\boldsymbol{\sigma}_H \\ C(\mu_{M,1}, \kappa_{M,1})\,\boldsymbol{\sigma}_{M,1} \\ \vdots \\ C(\mu_{M,L}, \kappa_{M,L})\,\boldsymbol{\sigma}_{M,L} \end{pmatrix}, \qquad P_2 w \;:=\; \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \omega_{\boldsymbol{\sigma},1}\boldsymbol{\sigma}_{M,1} \\ \vdots \\ \omega_{\boldsymbol{\sigma},L}\boldsymbol{\sigma}_{M,L} \end{pmatrix} \tag{5.30}$$

and $Q : X \supseteq \mathcal{D}(B) \to X$,

$$Qw \;:=\; \begin{pmatrix} \operatorname{div}\left(\boldsymbol{\sigma}_H + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l}\right) \\ \varepsilon(\mathbf{v}) \\ \varepsilon(\mathbf{v}) \\ \vdots \\ \varepsilon(\mathbf{v}) \end{pmatrix}, \tag{5.31}$$

$w = (\mathbf{v}, \boldsymbol{\sigma}_H, \boldsymbol{\sigma}_M)^\top \in \mathcal{D}(B)$.

Proof. For $w = (\mathbf{v}, \boldsymbol{\sigma}_H, \boldsymbol{\sigma}_M) \in \mathcal{D}(B)$ we simply compare Bw in (5.29) with $(-P_1 Q + P_2)\,w$ by a direct calculation. □

Lemma 60. *The linear operators* $P_1, P_2 : (X, \|.\|_X) \to (X, \|.\|_X)$ *defined in* (5.30) *are bounded.*

Proof. This follows from Lemma 51 together with $\vartheta, \ldots, \omega_{\boldsymbol{\sigma},L} \in L^\infty(D)$ as assumed in (5.13). □

Lemma 61. *The operator* P_1 *defined in* (5.30) *is self-adjoint, monotone and boundedly invertible with respect to* $(.\,,.)_X$.

Proof. The symmetry of P_1 follows from Lemma 50(h), the monotonicity from Lemma 50(i), the invertibility from Lemma 50(g), and the boundedness of P_1^{-1} from Lemma 50(j). Throughout we make use of (5.13). □

Lemma 62. *The scalar product*

$$\begin{aligned} (w_1, w_2)_E \;:=\; & \left(\frac{1}{\vartheta}\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\right)_{L^2(D,\mathbb{R}^3)} + \left(C\Big(\frac{1}{\mu_H}, \frac{1}{\kappa_H}\Big)\boldsymbol{\sigma}_H^{(1)}, \boldsymbol{\sigma}_H^{(2)}\right)_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})} \\ & + \sum_{l=1}^{L} \left(C\Big(\frac{1}{\mu_{M,l}}, \frac{1}{\kappa_{M,l}}\Big)\boldsymbol{\sigma}_{M,l}^{(1)}, \boldsymbol{\sigma}_{M,l}^{(2)}\right)_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})}, \end{aligned} \tag{5.32}$$

$w_i = \left(\mathbf{v}^{(i)}, \boldsymbol{\sigma}_H^{(i)}, \boldsymbol{\sigma}_M^{(i)}\right)^\top \in X, \quad i = 1, 2,$ *is well-defined and provides a norm* $\|.\|_E$ *on* X *which is equivalent to* $\|.\|_X$.

Proof. Since $(w_1, w_2)_E = (P_1^{-1} w_1, w_2)_X$, $w_1, w_2 \in X$, with P_1 as defined in (5.30), this statement follows from Lemma 61 and Lemma 36. $\square$

Lemma 63. *In analogy to section 4.1 we define the scalar product*

$$
\begin{aligned}
&(u_1, u_2)_T \\
&\quad \coloneqq \quad (Tu_1,\, Tu_2)_E \\
&\quad = \quad \Big(\frac{1}{\vartheta}\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\Big)_{L^2(D,\mathbb{R}^3)} \qquad (5.33)\\
&\quad\quad + \; \Big(C\Big(\frac{1}{\mu_H}, \frac{1}{\kappa_H}\Big)\Big(\boldsymbol{\sigma}^{(1)} + \sum_{l=1}^{L}\frac{1}{\omega_{\boldsymbol{\sigma},l}}\boldsymbol{\eta}_l^{(1)}\Big)\,,\, \boldsymbol{\sigma}^{(2)} + \sum_{l=1}^{L}\frac{1}{\omega_{\boldsymbol{\sigma},l}}\boldsymbol{\eta}_l^{(2)}\Big)_{L^2(D,\mathbb{R}^{3\times3}_{\text{sym}})} \\
&\quad\quad + \; \sum_{l=1}^{L}\Big(\frac{1}{\omega_{\boldsymbol{\sigma},l}^2}C\Big(\frac{1}{\mu_{M,l}}\,,\, \frac{1}{\kappa_{M,l}}\Big)\boldsymbol{\eta}_l^{(1)}\,,\, \boldsymbol{\eta}_l^{(2)}\Big)_{L^2(D,\mathbb{R}^{3\times3}_{\text{sym}})},
\end{aligned}
$$

for $u_i = \big(\mathbf{v}^{(i)}, \boldsymbol{\sigma}^{(i)}, \boldsymbol{\eta}^{(i)}\big)^\top \in X$, $\quad i = 1, 2.$ *The norm* $\|.\|_T$ *induced by it is equivalent to* $\|.\|_X$.

Proof. That $(.\,,.)_T$ and $(.\,,.)_X$ are equivalent is stated in Lemma 24. $\square$

Notation 64. Again like in section 4.1 we endow $\mathcal{D}(A)$ and $\mathcal{D}(B)$ with the graph norms $\|u\|_{A,T} \coloneqq \|u\|_T + \|Au\|_T$, $u \in \mathcal{D}(A)$, and $\|w\|_{B,E} \coloneqq \|w\|_E + \|Bw\|_E$, $w \in \mathcal{D}(B)$. $\square$

Theorem 65. *(Stress Decomposition)*
Let $\big(\mathbf{v}^{(0)}, \boldsymbol{\sigma}^{(0)}, \boldsymbol{\eta}^{(0)}\big)^\top \in \mathcal{D}(A)$, $\tilde{\mathbf{f}} : [0, t_1] \to L^2(D, \mathbb{R}^3)$ *and* $\mathbf{g} : [0, t_1] \to L^2(D, \mathbb{R}^{3\times3}_{\text{sym}})$. *A function* $(\mathbf{v}, \boldsymbol{\sigma}, \boldsymbol{\eta})^\top \in C^1\big([0, t_1], X\big) \cap C\big([0, t_1], \mathcal{D}(A)\big)$ *solves* (5.23), *iff* $(\mathbf{v}, \boldsymbol{\sigma}_H, \boldsymbol{\sigma}_M)^\top \in C^1\big([0, t_1], X\big) \cap C\big([0, t_1], \mathcal{D}(B)\big)$ *with* $\boldsymbol{\sigma}_H \coloneqq \boldsymbol{\sigma} + \sum_{l=1}^{L} \frac{1}{\omega_{\boldsymbol{\sigma},l}}\boldsymbol{\eta}_l$ *and* $\boldsymbol{\sigma}_{M,l} \coloneqq -\frac{1}{\omega_{\boldsymbol{\sigma},l}}\boldsymbol{\eta}_l$, $l = 1, \ldots, L$, *solves the transformed initial-boundary value problem*

$$
\begin{aligned}
\mathbf{v}'(t) \quad &= \quad \vartheta\,\mathrm{div}\Big(\boldsymbol{\sigma}_H + \sum_{l=1}^{L}\boldsymbol{\sigma}_{M,l}\Big)(t) \; + \; \tilde{\mathbf{f}}(t)\,, \\
\boldsymbol{\sigma}_H'(t) \quad &= \quad C(\mu_H\,,\, \kappa_H)\,\varepsilon\big(\mathbf{v}(t)\big) \; + \; \mathbf{g}(t)\,, \\
\boldsymbol{\sigma}_{M,l}'(t) \quad &= \quad C(\mu_{M,l}\,,\, \kappa_{M,l})\,\varepsilon\big(\mathbf{v}(t)\big) \; - \; \omega_{\boldsymbol{\sigma},l}\boldsymbol{\sigma}_{M,l}(t)\,, \qquad (5.34)
\end{aligned}
$$

$$
\mathbf{v}(0) \; = \; \mathbf{v}^{(0)}\,, \qquad \boldsymbol{\sigma}_H(0) \; = \; \boldsymbol{\sigma}^{(0)} + \sum_{l=1}^{L}\frac{1}{\omega_{\boldsymbol{\sigma},l}}\boldsymbol{\eta}_l^{(0)}\,, \qquad \boldsymbol{\sigma}_{M,l}(0) \; = \; -\frac{1}{\omega_{\boldsymbol{\sigma},l}}\boldsymbol{\eta}_l^{(0)}\,,
$$

$$
\mathbf{v}(t)\big|_{\partial D_D} \; = \; \mathbf{0}\,, \qquad \mathbf{n}^\top\Big(\boldsymbol{\sigma}_H + \sum_{l=1}^{L}\boldsymbol{\sigma}_{M,l}\Big)(t)\Big|_{\partial D_N} \; = \; \mathbf{0}\,,
$$

$l = 1, \ldots, L, \quad t \in [0, t_1].$

Proof. With $u := (\mathbf{v}, \boldsymbol{\sigma}, \boldsymbol{\eta})^\top$, $u_0 := (\mathbf{v}^{(0)}, \boldsymbol{\sigma}^{(0)}, \boldsymbol{\eta}^{(0)})^\top$,

$$Tu_0 \quad = \quad \Big(\mathbf{v}^{(0)},\, \boldsymbol{\sigma}^{(0)} + \sum_{l=1}^{L} \frac{1}{\omega_{\boldsymbol{\sigma},l}} \boldsymbol{\eta}_l^{(0)},\, -\frac{1}{\omega_{\boldsymbol{\sigma},l}} \boldsymbol{\eta}_1^{(0)},\, \ldots,\, -\frac{1}{\omega_{\boldsymbol{\sigma},L}} \boldsymbol{\eta}_L^{(0)}\Big)^\top,$$

$f := (\tilde{\mathbf{f}}, \mathbf{g}, \mathbf{0})^\top$ and $w := \tilde{T}u = (\mathbf{v}, \boldsymbol{\sigma}_H, \boldsymbol{\sigma}_M)^\top$, where $(\tilde{T}u)(t) = T\big(u(t)\big)$, $t \in [0, t_1]$, as in (4.3), we can write (5.34) as the evolution equation

$$w'(t) \;=\; -B\,w(t) \,+\, f(t)\,, \qquad t \in [0, t_1]\,, \qquad\qquad w(0) \;=\; Tu_0\,.$$

Since by definition of T in (5.24) it holds

$$T\big(f(t)\big) \;=\; T\big(\tilde{\mathbf{f}}(t), \mathbf{g}(t), \mathbf{0}\big)^\top \;=\; f(t)\,, \qquad\qquad t \in [0, t_1]\,,$$

this theorem is an application of Theorem 27. □

Remark 66. *The rheological model behind* (5.23) *is the Generalized Maxwell body with one Hooke and L Maxwell elements connected in parallel which is also known as Maxwell-Wiechert model (see [13] for example). It is diagrammatically illustrated in Figure 5.1.*

A Hooke element can be thought of as an ideal spring. A Maxwell element consists of a Hooke element and a Newton element, which is also called a dashpot, connected in series.

For two such elements connected in series the strains of both sum up whereas the stresses are equal in each element. For two elements connected in parallel in turn the stresses sum up whereas the strains are equal.

As in chapter 2 we denote the overall displacement vector by $\mathbf{u}$*, such that the overall velocity* $\mathbf{v}$ *is given by* $\partial\mathbf{u}/\partial t$*. Then the overall strain is described by* $\varepsilon(\mathbf{u})$*. By* $\mathbf{u}_{H,1}, \ldots, \mathbf{u}_{H,L}$ *and* $\mathbf{u}_{N,1}, \ldots, \mathbf{u}_{N,L}$ *and* $\mathbf{v}_{H,1}, \ldots, \mathbf{v}_{H,L}$ *and* $\mathbf{v}_{N,1}, \ldots, \mathbf{v}_{N,L}$ *we denote the displacements and velocities of the Hooke and Newton parts of the individual Maxwell elements, respectively. The variables* $\boldsymbol{\sigma}_{M,1}, \ldots, \boldsymbol{\sigma}_{M,L}$ *denote the*

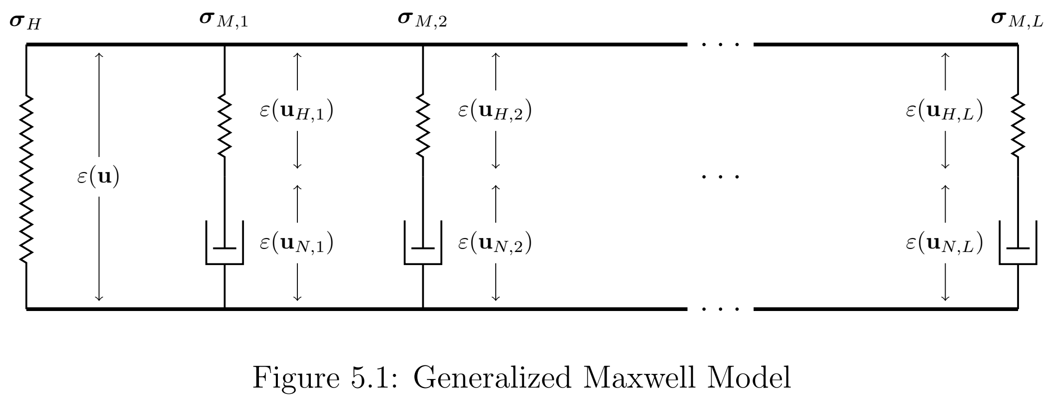

Figure 5.1: Generalized Maxwell Model

stresses of the Maxwell elements, the variable $\boldsymbol{\sigma}_H$ *denotes the sum of the stress of the single Hooke element and the external stress* $\tilde{\mathbf{g}}$. *(Note, that* $\mathbf{g} = \partial\tilde{\mathbf{g}}/\partial t$, *as introduced in section 2.2.) Furthermore,* $\mu_{M,1}, \ldots, \mu_{M,L}$ *and* $\kappa_{M,1}, \ldots, \kappa_{M,L}$ *are the shear and the bulk moduli of the Hooke elements which are part of the Maxwell elements. The shear and bulk modulus of the material related to the single Hooke element is given by* μ_H *and* κ_H.

For a Hooke element, stress depends linearly on strain. More precisely, the Hooke elements which are part of the individual Maxwell elements, are characterized by

$$\boldsymbol{\sigma}_{M,l} \;=\; C(\mu_{M,l}\,,\,\kappa_{M,l})\,\varepsilon\big(\mathbf{u}_{H,l}\big)\,, \qquad\qquad l = 1, \ldots, L\,. \tag{5.35}$$

For a Newton element in turn, stress depends linearly on strain velocity. That is, the Newton element parts of the individual Maxwell elements have the property

$$\boldsymbol{\sigma}_{M,l} \;=\; \frac{1}{\omega_{\boldsymbol{\sigma},l}} C(\mu_{M,l}\,,\,\kappa_{M,l})\,\varepsilon(\mathbf{v}_{N,l})\,, \qquad\qquad l = 1, \ldots, L\,. \tag{5.36}$$

Moreover, the stress arising from the single Hooke element and the external stress $\tilde{\mathbf{g}}$ *are subsumed to*

$$\boldsymbol{\sigma}_H \;=\; C(\mu_H, \kappa_H)\,\varepsilon(\mathbf{u}) \;+\; \tilde{\mathbf{g}}\,. \tag{5.37}$$

From (5.35) – (5.37) *we derive the second and third through* $(2{+}L)$*th equation of* (5.34) *in the following way: The second equation of* (5.34) *is the time derivative of* (5.37). *For each* $l \in \{1, \ldots, L\}$, *the corresponding equation of* (5.34) *is the sum of the time derivative of* (5.35) *with* $\omega_{\boldsymbol{\sigma},l}$ *times* (5.36). *Here, we also use the connections mentioned above,*

$$\mathbf{u} \;=\; \mathbf{u}_{H,l} + \mathbf{u}_{N,l}\,, \qquad\qquad l = 1, \ldots, L\,.$$

Finally, the first equation of (5.34) *is Newton's second law:*

$$\rho\,\mathbf{v}' \;=\; \operatorname{div}\Big(\boldsymbol{\sigma}_H + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l}\Big) \;+\; \mathbf{f}\,,$$

which reads

$$\mathbf{v}' \;=\; \vartheta \operatorname{div}\Big(\boldsymbol{\sigma}_H + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l}\Big) \;+\; \tilde{\mathbf{f}}$$

in our notation $\vartheta = 1/\rho$ *and* $\tilde{\mathbf{f}} = \mathbf{f}/\rho$. □

Remark 67. *With a look at Lemma 50*(k) *we see*[2] *that it would also be possible to incorporate different stress relaxation times* $\tau_{\boldsymbol{\sigma},\mathrm{S},l}$ *and* $\tau_{\boldsymbol{\sigma},\mathrm{P},l}$ *with reciprocals* $\omega_{\boldsymbol{\sigma},\mathrm{S},l}$

[2]Many thanks to Elena Cherkaev for a very inspiring discussion which led to this idea.

and $\omega_{\boldsymbol{\sigma},\mathrm{P},l}$ *for the shear and the bulk parts of the stress components, respectively, by substituting the last equation in* (5.34) *by*

$$\boldsymbol{\sigma}'_{M,l}(t) \;=\; C(\mu_{M,l}\,,\,\kappa_{M,l})\,\varepsilon\big(\mathbf{v}(t)\big) \;-\; C(\omega_{\boldsymbol{\sigma},\mathrm{S},l}\,,\,\omega_{\boldsymbol{\sigma},\mathrm{P},l})\,\boldsymbol{\sigma}_{M,l}(t)\,, \qquad l=1,\dots,L\,,$$

$t \in [0,t_1]$. □

Remark 68. *For each state* $w = (\mathbf{v},\boldsymbol{\sigma}_H,\boldsymbol{\sigma}_M)^\top \in \mathcal{D}(B)$ *of* (5.34)*, the value of the mechanical energy of the physical system described by* (5.34) *is given by*

$$\begin{aligned}\frac{1}{2}\|w\|_E^2 \;&=\; \frac{1}{2}\Big(\frac{1}{\vartheta}\mathbf{v}\,,\,\mathbf{v}\Big)_{L^2(D,\mathbb{R}^3)} \;+\; \frac{1}{2}\Big(C(\mu_H,\kappa_H)^{-1}\boldsymbol{\sigma}_H\,,\,\boldsymbol{\sigma}_H\Big)_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})}\\ &\quad+\;\frac{1}{2}\sum_{l=1}^{L}\Big(C(\mu_{M,l}\,,\,\kappa_{M,l})^{-1}\boldsymbol{\sigma}_{M,l}\,,\,\boldsymbol{\sigma}_{M,l}\Big)_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})}\,.\end{aligned}$$

In the case $\mathbf{g}=\mathbf{0}$*, this is the sum of the kinetic energy and the strain energy stored within the individual Hooke elements.*

In the original variables, the mechanical energy of a state $u = (\mathbf{v},\boldsymbol{\sigma},\boldsymbol{\eta})^\top \in \mathcal{D}(A)$ *of the physical system described by the initial-boundary value problem* (5.23) *has the form*

$$\begin{aligned}\frac{1}{2}\|u\|_T^2 \;&=\; \frac{1}{2}\Big(\frac{1}{\vartheta}\mathbf{v},\mathbf{v}\Big)_{L^2(D,\mathbb{R}^3)}\\ &\quad+\;\frac{1}{2}\left(C(\mu_H,\kappa_H)^{-1}\Big(\boldsymbol{\sigma}+\sum_{l=1}^{L}\frac{1}{\omega_{\boldsymbol{\sigma},l}}\boldsymbol{\eta}_l\Big)\,,\,\boldsymbol{\sigma}+\sum_{l=1}^{L}\frac{1}{\omega_{\boldsymbol{\sigma},l}}\boldsymbol{\eta}_l\right)_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})}\\ &\quad+\;\frac{1}{2}\sum_{l=1}^{L}\Big(\frac{1}{\omega_{\boldsymbol{\sigma},l}^2}C(\mu_{M,l}\,,\,\kappa_{M,l})^{-1}\boldsymbol{\eta}_l\,,\,\boldsymbol{\eta}_l\Big)_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})}\,.\end{aligned}$$

For $u = T^{-1}w$*, Lemma 25 assures that both are equal.* □

5.4 Existence, Uniqueness, Energy Balance

To prove existence and uniqueness of the solution of (5.23) in the function space $C^1\big([0,t_1],X\big)\cap C\big([0,t_1],\mathcal{D}(A)\big)$, we use Theorem 65 and apply the results of sections 3.1 and 3.2 to (5.34).

To show that the operator B in (5.29) generates a contraction semigroup, we make use of Lemma 8 in Section 3.1, which is a consequence of the Theorem of Hille-Yosida (Theorem 3 and Theorem 4). Consequently we are going to prove that B is maximal monotone in the sense of Definition 1 with respect to the scalar product $(.\,,.)_E$ defined in (5.32).

Proposition 69. *The operator B from (5.29) is monotone in the sense of (3.3) with respect to the scalar product $(.\,,.)_E$ defined in (5.32). That is, $(Bw, w)_E \geq 0$, $w \in \mathcal{D}(B)$, where $\mathcal{D}(B)$ is of the form (5.28).*

Proof. For $w = (\mathbf{v}, \boldsymbol{\sigma}_H, \boldsymbol{\sigma}_M) \in \mathcal{D}(B)$ it is $\mathbf{v} \in V$ and $(\boldsymbol{\sigma}_H + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l}) \in S$. So from (5.10) and Lemma 50 (g) and (i) it follows

$$
\begin{aligned}
(Bw, w)_E \quad &= \quad \int_D - \operatorname{div}\Big(\boldsymbol{\sigma}_H + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l}\Big) \cdot \mathbf{v} \; - \; \varepsilon(\mathbf{v}) : \boldsymbol{\sigma}_H \\
&\qquad - \sum_{l=1}^{L} \Big(\varepsilon(\mathbf{v}) - \omega_{\boldsymbol{\sigma},l} C\Big(\frac{1}{\mu_{M,l}}\,,\,\frac{1}{\kappa_{M,l}}\Big)\boldsymbol{\sigma}_{M,l}\Big) : \boldsymbol{\sigma}_{M,l}\, dx \\
&= \quad -\int_D \varepsilon(\mathbf{v}) : \Big(\boldsymbol{\sigma}_H + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l}\Big) \; + \; \mathbf{v} \cdot \operatorname{div}\Big(\boldsymbol{\sigma}_H + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l}\Big) \\
&\qquad - \sum_{l=1}^{L} \omega_{\boldsymbol{\sigma},l} C\Big(\frac{1}{\mu_{M,l}}\,,\,\frac{1}{\kappa_{M,l}}\Big)\boldsymbol{\sigma}_{M,l} : \boldsymbol{\sigma}_{M,l}\, dx \\
&= \quad \int_D \sum_{l=1}^{L} \omega_{\boldsymbol{\sigma},l} C\Big(\frac{1}{\mu_{M,l}}\,,\,\frac{1}{\kappa_{M,l}}\Big)\boldsymbol{\sigma}_{M,l} : \boldsymbol{\sigma}_{M,l}\, dx \\
&\geq \quad \int_D \sum_{l=1}^{L} \omega_{\boldsymbol{\sigma},l} \min\Big\{\frac{1}{\mu_{M,l}}\,,\,\frac{1}{\kappa_{M,l}}\Big\}\boldsymbol{\sigma}_{M,l} : \boldsymbol{\sigma}_{M,l}\, dx \\
&\geq \quad 0\,.
\end{aligned}
$$

□

Proposition 70. *With B from (5.29) and $\mathcal{D}(B)$ as in (5.28), the operator*

$$\mathrm{Id} + B \; : \quad \mathcal{D}(B) \;\to\; X$$

is onto.

Proof. Let $f \in X$. We need to prove the existence of an element $w \in \mathcal{D}(B)$ such that $(\mathrm{Id} + B)\, w = f$. With $f = (\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_{3,1}, \ldots, \boldsymbol{f}_{3,L})^\top$ and $w = (\mathbf{v}, \boldsymbol{\sigma}_H, \boldsymbol{\sigma}_M)^\top$ this explicitly reads

$$
\begin{aligned}
\mathbf{v} \; - \; \vartheta \operatorname{div}\Big(\boldsymbol{\sigma}_H + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l}\Big) \quad &= \quad \boldsymbol{f}_1\,, \\
\boldsymbol{\sigma}_H \; - \; C(\mu_H, \kappa_H)\, \varepsilon(\mathbf{v}) \quad &= \quad \boldsymbol{f}_2\,, \\
\boldsymbol{\sigma}_{M,l} \; - \; C(\mu_{M,l}\,, \kappa_{M,l})\, \varepsilon(\mathbf{v}) \; + \; \omega_{\boldsymbol{\sigma},l} \boldsymbol{\sigma}_{M,l} \quad &= \quad \boldsymbol{f}_{3,l}\,, \qquad\qquad l = 1, \ldots, L\,.
\end{aligned}
$$

Solving the second equation for $\boldsymbol{\sigma}_H$ and the equations in the third line for $\boldsymbol{\sigma}_{M,l}$ yields the equivalent system

$$\begin{aligned}
\mathbf{v} \;-\; \vartheta\,\mathrm{div}\Big(\boldsymbol{\sigma}_H + \sum_{l=1}^{L}\boldsymbol{\sigma}_{M,l}\Big) &= \boldsymbol{f}_1\,, && (5.38)\\
\boldsymbol{\sigma}_H &= \boldsymbol{f}_2 \;+\; C(\mu_H,\kappa_H)\,\varepsilon(\mathbf{v})\,, && (5.39)\\
\boldsymbol{\sigma}_{M,l} &= \frac{1}{1+\omega_{\boldsymbol{\sigma},l}}\big(\boldsymbol{f}_{3,l} + C(\mu_{M,l}\,,\,\kappa_{M,l})\,\varepsilon(\mathbf{v})\big)\,, && (5.40)
\end{aligned}$$

$l = 1,\dots,L$. To derive the weak formulation of the first equation, we multiply both sides by $1/\vartheta$ and a test function $\boldsymbol{\varphi}\in V$ and integrate over D. This gives

$$\int_D \frac{1}{\vartheta}\mathbf{v}\cdot\boldsymbol{\varphi} \;-\; \mathrm{div}\Big(\boldsymbol{\sigma}_H + \sum_{l=1}^{L}\boldsymbol{\sigma}_{M,l}\Big)\cdot\boldsymbol{\varphi}\,dx \quad=\quad \int_D \frac{1}{\vartheta}\,\boldsymbol{f}_1\cdot\boldsymbol{\varphi}\,dx\,, \qquad (5.41)$$

$\boldsymbol{\varphi}\in V$. Because V is dense in $L^2(D,\mathbb{R}^3)$, which was proven in Lemma 46, equation (5.38) and equation (5.41) are equivalent.

Since we assumed $(\boldsymbol{\varphi},\boldsymbol{\sigma}_H,\boldsymbol{\sigma}_M)^\top\in\mathcal{D}(B)$, it follows from (5.28) and (5.10) that a partial integration of the second summand on the left-hand side of (5.41) results in

$$\int_D \frac{1}{\vartheta}\mathbf{v}\cdot\boldsymbol{\varphi} \;+\; \Big(\boldsymbol{\sigma}_H + \sum_{l=1}^{L}\boldsymbol{\sigma}_{M,l}\Big) : \varepsilon(\boldsymbol{\varphi})\,dx \quad=\quad \int_D \frac{1}{\vartheta}\,\boldsymbol{f}_1\cdot\boldsymbol{\varphi}\,dx\,,$$

$\boldsymbol{\varphi}\in V$.

Now we plug in the second equation and the equations in the third line into this weak form of the first one and get

$$\begin{aligned}
&\int_D \frac{1}{\vartheta}\mathbf{v}\cdot\boldsymbol{\varphi} \;+\; \Big(\big(\boldsymbol{f}_2 \;+\; C(\mu_H,\kappa_H)\,\varepsilon(\mathbf{v})\big)\\
&\qquad\qquad + \sum_{l=1}^{L}\frac{1}{1+\omega_{\boldsymbol{\sigma},l}}\big(\boldsymbol{f}_{3,l} + C(\mu_{M,l}\,,\,\kappa_{M,l})\,\varepsilon(\mathbf{v})\big)\Big) : \varepsilon(\boldsymbol{\varphi})\,dx \qquad (5.42)\\
&\quad= \int_D \frac{1}{\vartheta}\,\boldsymbol{f}_1\cdot\boldsymbol{\varphi}\,dx\,,
\end{aligned}$$

$\boldsymbol{\varphi}\in V$. After rearranging the terms, this equation takes the form

$$\begin{aligned}
&\int_D \frac{1}{\vartheta}\mathbf{v}\cdot\boldsymbol{\varphi} \;+\; \Big(C(\mu_H,\kappa_H) + \sum_{l=1}^{L}\frac{1}{1+\omega_{\boldsymbol{\sigma},l}}C(\mu_{M,l}\,,\,\kappa_{M,l})\Big)\varepsilon(\mathbf{v}) : \varepsilon(\boldsymbol{\varphi})\,dx\\
&\quad= \int_D \frac{1}{\vartheta}\,\boldsymbol{f}_1\cdot\boldsymbol{\varphi} \;-\; \Big(\boldsymbol{f}_2 + \sum_{l=1}^{L}\frac{1}{1+\omega_{\boldsymbol{\sigma},l}}\boldsymbol{f}_{3,l}\Big) : \varepsilon(\boldsymbol{\varphi})\,dx\,,
\end{aligned} \qquad (5.43)$$

$\boldsymbol{\varphi} \in V$.

In order to prove the existence of a unique solution of this equation, we are going to use the Lax-Milgram theorem (see [11], section 6.2.1, Theorem 1 for example).

The left-hand side of (5.43) can be understood as a bounded bilinearform $V \times V \to \mathbb{R}$ in the variables $\mathbf{v}$ and $\boldsymbol{\varphi}$ with respect to the scalar product $(.\,,.)_V$ defined in (5.8), since with Lemma 50 (a) and (e), Lemma 51, the parameter bounds prescribed in Assumption 49 and the Cauchy-Schwarz inequality, it holds

$$\begin{aligned}
&\left| \int_D \frac{1}{\vartheta} \mathbf{v} \cdot \boldsymbol{\varphi} \; + \; \Big(C(\mu_H, \kappa_H) + \sum_{l=1}^{L} \frac{1}{1+\omega_{\boldsymbol{\sigma},l}} C(\mu_{M,l}\,,\, \kappa_{M,l}) \Big) \varepsilon(\mathbf{v}) : \varepsilon(\boldsymbol{\varphi}) \; dx \right| \\
\leq \quad & \Big\| \frac{1}{\vartheta} \Big\|_{L^\infty(D)} \|\mathbf{v}\|_{L^2(D,\mathbb{R}^3)} \, \|\boldsymbol{\varphi}\|_{L^2(D,\mathbb{R}^3)} \\
& \quad + \; \max \left\{ \Big\| \mu_H + \sum_{l=1}^{L} \frac{\mu_{M,l}}{1+\omega_{\boldsymbol{\sigma},l}} \Big\|_{L^\infty(D)} , \; \Big\| \kappa_H + \sum_{l=1}^{L} \frac{\kappa_{M,l}}{1+\omega_{\boldsymbol{\sigma},l}} \Big\|_{L^\infty(D)} \right\} \\
& \qquad \qquad \qquad \big\| \varepsilon(\mathbf{v}) \big\|_{L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}})} \, \big\| \varepsilon(\boldsymbol{\varphi}) \big\|_{L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}})} \\
\leq \quad & \max \left\{ \Big\| \frac{1}{\vartheta} \Big\|_{L^\infty(D)} , \; \Big\| \mu_H + \sum_{l=1}^{L} \frac{\mu_{M,l}}{1+\omega_{\boldsymbol{\sigma},l}} \Big\|_{L^\infty(D)} , \; \Big\| \kappa_H + \sum_{l=1}^{L} \frac{\kappa_{M,l}}{1+\omega_{\boldsymbol{\sigma},l}} \Big\|_{L^\infty(D)} \right\} \\
& \qquad \qquad \qquad \|\mathbf{v}\|_V \, \|\boldsymbol{\varphi}\|_V \, ,
\end{aligned}$$

$\mathbf{v}, \boldsymbol{\varphi} \in V$. This bilinearform is even bounded from below, since with Lemma 50 (a), (e), (i) and the parameter bounds given by Assumption 49, it holds

$$\begin{aligned}
&\int_D \frac{1}{\vartheta} \mathbf{v} \cdot \mathbf{v} \; + \; \Big(C(\mu_H, \kappa_H) + \sum_{l=1}^{L} \frac{1}{1+\omega_{\boldsymbol{\sigma},l}} C(\mu_{M,l}\,,\, \kappa_{M,l}) \Big) \varepsilon(\mathbf{v}) : \varepsilon(\mathbf{v}) \; dx \\
\geq \quad & \frac{1}{\|\vartheta\|_{L^\infty(D)}} \, (\mathbf{v}, \mathbf{v})_{L^2(D,\mathbb{R}^3)} \\
& \quad + \; \min \Big\{ \mu_{H,0} + \sum_{l=1}^{L} \frac{\mu_{M,l,0}}{1 + \|\omega_{\boldsymbol{\sigma},l}\|_{L^\infty(D)}} \, , \\
& \qquad \qquad \kappa_{H,0} + \sum_{l=1}^{L} \frac{\kappa_{M,l,0}}{1 + \|\omega_{\boldsymbol{\sigma},l}\|_{L^\infty(D)}} \Big\} \, \big(\varepsilon(\mathbf{v}), \varepsilon(\mathbf{v}) \big)_{L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}})} \\
\geq \quad & \min \Big\{ \frac{1}{\|\vartheta\|_{L^\infty(D)}} \, , \\
& \qquad \mu_{H,0} + \sum_{l=1}^{L} \frac{\mu_{M,l,0}}{1 + \|\omega_{\boldsymbol{\sigma},l}\|_{L^\infty(D)}} \, , \; \kappa_{H,0} + \sum_{l=1}^{L} \frac{\kappa_{M,l,0}}{1 + \|\omega_{\boldsymbol{\sigma},l}\|_{L^\infty(D)}} \Big\} \, (\mathbf{v}, \mathbf{v})_V \, ,
\end{aligned}$$

$\mathbf{v} \in V$.

The right-hand side of (5.43) in turn can be interpreted as a bounded linear functional on $(V, \|.\|_V)$ in the variable $\boldsymbol{\varphi}$, since

$$\begin{aligned}
&\left| \int_D \frac{1}{\vartheta} \boldsymbol{f}_1 \cdot \boldsymbol{\varphi} \;-\; \Big(\boldsymbol{f}_2 + \sum_{l=1}^{L} \frac{1}{1+\omega_{\boldsymbol{\sigma},l}} \boldsymbol{f}_{3,l} \Big) : \varepsilon(\boldsymbol{\varphi}) \, dx \right| \\
&\qquad \leq \quad \frac{1}{\vartheta_0} \|\boldsymbol{f}_1\|_{L^2(D,\mathbb{R}^3)} \|\boldsymbol{\varphi}\|_{L^2(D,\mathbb{R}^3)} \\
&\qquad\qquad + \; \Big\| \boldsymbol{f}_2 + \sum_{l=1}^{L} \frac{1}{1+\omega_{\boldsymbol{\sigma},l}} \boldsymbol{f}_{3,l} \Big\|_{L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}})} \big\| \varepsilon(\boldsymbol{\varphi}) \big\|_{L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}})} \\
&\qquad \leq \quad \sqrt{ \frac{1}{\vartheta_0^2} \|\boldsymbol{f}_1\|^2_{L^2(D,\mathbb{R}^3)} \; + \; \Big\| \boldsymbol{f}_2 + \sum_{l=1}^{L} \frac{1}{1+\omega_{\boldsymbol{\sigma},l}} \boldsymbol{f}_{3,l} \Big\|^2_{L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}})} } \quad \|\boldsymbol{\varphi}\|_V
\end{aligned}$$

$\boldsymbol{\varphi} \in V$.

So by the Lax-Milgram theorem, there exists a unique solution $\mathbf{v}$ of (5.43). With this distinct $\mathbf{v}$, we define $\boldsymbol{\sigma}_H$ by (5.39) and $\boldsymbol{\sigma}_{M,l}$ by (5.40), $l = 1, \ldots, L$.

To see that $(\mathbf{v}, \boldsymbol{\sigma}_H, \boldsymbol{\sigma}_M)^\top$ is an element of $\mathcal{D}(B)$, we consider equation (5.42), which is equivalent to equation (5.43). In (5.42) we reversely substitute the terms on the right-hand sides of (5.39) and (5.40) by $\boldsymbol{\sigma}_H$ and $\boldsymbol{\sigma}_{M,l}$, respectively. After subtraction of the first term on the left-hand side, this yields

$$\int_D \Big(\boldsymbol{\sigma}_H + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l} \Big) : \varepsilon(\boldsymbol{\varphi}) \, dx \quad = \quad - \int_D \frac{1}{\vartheta} (\mathbf{v} - \boldsymbol{f}_1) \cdot \boldsymbol{\varphi} \, dx \,, \tag{5.44}$$

$\boldsymbol{\varphi} \in V$. Since this statement in particular holds for all $\boldsymbol{\varphi} \in C_c^\infty(D, \mathbb{R}^3)$, the term $(1/\vartheta)\,(\mathbf{v} - \boldsymbol{f}_1) \in L^2(D, \mathbb{R}^3)$ by definition is $\operatorname{div}\big(\boldsymbol{\sigma}_H + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l}\big)$ in weak form. Now (5.44) explicitly assures, that $\boldsymbol{\sigma}_H + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l}$ satisfies the variationally formulated boundary condition in (5.10).

Thus $\boldsymbol{\sigma}_H + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l} \in S$, $w \coloneqq (\mathbf{v}, \boldsymbol{\sigma}_H, \boldsymbol{\sigma}_M)^\top \in \mathcal{D}(B)$ and $(\mathrm{Id} + B)w = f$. This completes the proof. □

Corollary 71. *The operator* $B : X \supseteq \mathcal{D}(B) \to X$ *from* (5.29) *with* $\mathcal{D}(B)$ *from* (5.28) *is maximal monotone in the sense of Definition 1 in section 3.1 with respect to the scalar product* $(.\,,.)_E$ *from* (5.32)*.*

Proof. This corollary only subsumes Proposition 69 and Proposition 70. □

Now we are able to prove existence and uniqueness of the solution of the transformed initial-boundary value problem (5.34). For the convenience of the reader, we repeat the spaces in use.

Theorem 72. *Let* $\vartheta, \mu_H, \kappa_H, \mu_{M,l}, \kappa_{M,l}, \omega_{\boldsymbol{\sigma},l} \in L_+^\infty(D)$, $l = 1, \ldots, L$, *where*

$$L_+^\infty(D) \;=\; \Big\{ \alpha \in L^\infty(D) \,:\, \alpha > 0,\; \frac{1}{\alpha} \in L^\infty(D) \Big\} \tag{5.45}$$

as defined in (5.13), *and let*

$$X \quad = \quad L^2(D, \mathbb{R}^3) \times L^2(D, \mathbb{R}^{3\times 3}_{\mathrm{sym}}) \times L^2(D, \mathbb{R}^{3\times 3}_{\mathrm{sym}})^L \tag{5.46}$$

be the Hilbert space from (5.4). *Let*

$$\mathcal{D}(B) \quad = \quad \Big\{ (\mathbf{v}, \boldsymbol{\sigma}_H, \boldsymbol{\sigma}_M)^\top \in\; V \times L^2(D, \mathbb{R}^{3\times 3}_{\mathrm{sym}}) \times L^2(D, \mathbb{R}^{3\times 3}_{\mathrm{sym}})^L \;: \; \boldsymbol{\sigma}_H + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l} \in S \Big\}$$

as in (5.28), *with*

$$V \;=\; \overline{\Big\{\; \boldsymbol{\varphi} \in C^\infty(D, \mathbb{R}^3) \cap H(\varepsilon, D, \mathbb{R}^3) \;:\quad \partial D_D \subseteq \mathbb{R}^3 \setminus \operatorname{supp}(\boldsymbol{\varphi}) \;\Big\}}^{\|\cdot\|_V} \tag{5.47}$$

as in (5.9), *where the closure is taken in* $H(\varepsilon, D, \mathbb{R}^3)$, *and*

$$S \;=\; \Big\{\; \boldsymbol{\sigma} \in H(\operatorname{div}, D, \mathbb{R}^{3\times 3}_{\mathrm{sym}}) :\; \forall \boldsymbol{\varphi} \in V :\, \int_D \varepsilon(\boldsymbol{\varphi}) : \boldsymbol{\sigma} + \boldsymbol{\varphi} \cdot \operatorname{div} \boldsymbol{\sigma} \, dx = 0 \;\Big\} \tag{5.48}$$

as in (5.10), *and let* $\mathcal{D}(B)$ *be equipped with the graph norm* $\|\cdot\|_B = \|B\cdot\|_X + \|\cdot\|_X$. *Furthermore, let* $(\mathbf{v}^{(0)}, \boldsymbol{\sigma}_H^{(0)}, \boldsymbol{\sigma}_M^{(0)})^\top \in \mathcal{D}(B)$, $t_1 > 0$, $\mathbf{f} \in W^{1,1}\big((0,t_1), L^2(D,\mathbb{R}^3)\big)$, $\tilde{\mathbf{f}} = \vartheta \mathbf{f}$, *and* $\mathbf{g} \in W^{1,1}\big((0,t_1), L^2(D, \mathbb{R}^{3\times 3}_{\mathrm{sym}})\big)$. *Then the initial-boundary value problem*

$$\begin{aligned}
\mathbf{v}'(t) \quad &= \quad \vartheta \operatorname{div} \Big(\boldsymbol{\sigma}_H + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l} \Big)(t) \;+\; \tilde{\mathbf{f}}(t)\,, \\
\boldsymbol{\sigma}_H'(t) \quad &= \quad C(\mu_H\,,\, \kappa_H)\, \varepsilon\big(\mathbf{v}(t)\big) \;+\; \mathbf{g}(t)\,, \\
\boldsymbol{\sigma}_{M,l}'(t) \quad &= \quad C(\mu_{M,l}\,,\, \kappa_{M,l})\, \varepsilon\big(\mathbf{v}(t)\big) \;-\; \omega_{\boldsymbol{\sigma},l} \boldsymbol{\sigma}_{M,l}(t)\,, \qquad l = 1, \ldots, L\,,
\end{aligned} \tag{5.49}$$

$$\mathbf{v}(0) \;=\; \mathbf{v}^{(0)}\,, \qquad \boldsymbol{\sigma}_H(0) \;=\; \boldsymbol{\sigma}_H^{(0)}\,, \qquad \boldsymbol{\sigma}_M(0) \;=\; \boldsymbol{\sigma}_M^{(0)}\,,$$

$$\mathbf{v}(t)\big|_{\partial D_D} \;=\; \mathbf{0}\,, \qquad\qquad \mathbf{n}^\top \Big(\boldsymbol{\sigma}_H + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l} \Big)(t) \Big|_{\partial D_N} \;=\; \mathbf{0}\,,$$

$t \in [0, t_1]$, *has a unique solution* $(\mathbf{v}, \boldsymbol{\sigma}_H, \boldsymbol{\sigma}_M)^\top \in C^1\big([0,t_1], X\big) \cap C\big([0,t_1], \mathcal{D}(B)\big)$.

Proof. This theorem is a direct application of Corollary 18 in section 3.2 together with Corollary 71. □

Theorem 73. *The energy of the solution* $w = (\mathbf{v}, \boldsymbol{\sigma}_H, \boldsymbol{\sigma}_M)^\top$ *of the homogeneous version of* (5.49) *with* $\tilde{\mathbf{f}} = \mathbf{0}$ *and* $\mathbf{g} = \mathbf{0}$ *has the time derivative*

$$\begin{aligned} &\frac{d}{dt}\,\frac{1}{2}\|w(t)\|_E^2 \\ &= \quad -\int_D \sum_{l=1}^{L} \omega_{\boldsymbol{\sigma},l}(x) C\Big(\frac{1}{\mu_{M,l}(x)}\,,\,\frac{1}{\kappa_{M,l}(x)}\Big)\boldsymbol{\sigma}_{M,l}(x,t) : \boldsymbol{\sigma}_{M,l}(x,t)\,dx \\ &\leq \quad 0\,, \end{aligned} \tag{5.50}$$

$t \in [0, t_1]$. *In particular it holds*

$$\|w(t)\|_E \quad \leq \quad \|w_0\|_E\,, \qquad\qquad t \in [0, t_1]\,,$$

where $w_0 := \big(\mathbf{v}^{(0)}, \boldsymbol{\sigma}_H^{(0)}, \boldsymbol{\sigma}_M^{(0)}\big)^\top$.

Proof. This is an application of Theorem 4 in section 3.1, since according to the calculation in the proof of Proposition 69, the right-hand side of (5.50) is equal to $-(Bw(t), w(t))_E$. □

Remark 74. *With a look at* (5.50)*, for a solution* $w = (\mathbf{v}, \boldsymbol{\sigma}_H, \boldsymbol{\sigma}_M)^\top$ *of the homogeneous version of* (5.49) *with* $\tilde{\mathbf{f}} = \mathbf{0}$ *and* $\mathbf{g} = \mathbf{0}$*, the expression*

$$\sum_{l=1}^{L} \omega_{\boldsymbol{\sigma},l} C\Big(\frac{1}{\mu_{M,l}}\,,\,\frac{1}{\kappa_{M,l}}\Big)\boldsymbol{\sigma}_{M,l} : \boldsymbol{\sigma}_{M,l}$$

can be interpreted as the spatial density of the energy dissipation rate. □

Next, we turn to the original initial-boundary value problem (5.23).

Proposition 75. *The operator* $A : X \supseteq \mathcal{D}(A) \to X$ *from* (5.3) *with* $\mathcal{D}(A)$ *from* (5.11) *is maximal monotone in the sense of Definition 1 in section 3.1 with respect to the scalar product* $(\,.\,,.\,)_T$ *from* (5.33).

Proof. This is a consequence of Corollary 71 and Theorem 28 in section 4.1. □

Lemma 76. *The subspace* $\mathcal{D}(A)$ *defined in* (5.11) *is dense in* X*. In particular,* S *from* (5.10) *is dense in* $L^2(D, \mathbb{R}^{3\times3}_{\mathrm{sym}})$ *in analogy to the denseness of* V *from* (5.9) *in* $L^2(D, \mathbb{R}^3)$*, which has already been proven in Lemma 46.*

Proof. This follows from Proposition 75 and Lemma 2(a) in section 3.1. □

Theorem 77. *Let* $\vartheta, \mu_H, \kappa_H, \mu_{M,l}, \kappa_{M,l}, \omega_{\boldsymbol{\sigma},l} \in L^\infty_+(D)$, $l = 1, \ldots, L$, *as in* (5.45), *and let* X *be the Hilbert space* (5.46). *Let*

$$\mathcal{D}(A) \quad = \quad V \times S \times L^2(D, \mathbb{R}^{3\times 3}_{\mathrm{sym}})^L$$

from (5.11) *with* V *and* S *as in* (5.47) *and* (5.48) *be endowed with the graph norm* $\|\cdot\|_A = \|A\cdot\|_X + \|\cdot\|_X$, *and let* $\big(\mathbf{v}^{(0)}, \boldsymbol{\sigma}^{(0)}, \boldsymbol{\eta}^{(0)}\big)^\top \in \mathcal{D}(A)$. *Finally, let* $t_1 > 0$, $\mathbf{f} \in W^{1,1}\big((0,t_1), L^2(D,\mathbb{R}^3)\big)$, $\tilde{\mathbf{f}} = \vartheta\mathbf{f}$, *and* $\mathbf{g} \in W^{1,1}\big((0,t_1), L^2(D, \mathbb{R}^{3\times 3}_{\mathrm{sym}})\big)$. *Then the initial-boundary value problem*

$$\begin{aligned}
\mathbf{v}'(t) \quad &= \quad \vartheta \operatorname{div} \boldsymbol{\sigma}(t) \;+\; \tilde{\mathbf{f}}(t)\,,\\
\boldsymbol{\sigma}'(t) \quad &= \quad C\Big(\mu_H + \sum_{l=1}^{L} \mu_{M,l}\,,\; \kappa_H + \sum_{l=1}^{L}\kappa_{M,l}\Big)\,\varepsilon\big(\mathbf{v}(t)\big) \;+\; \sum_{l=1}^{L}\boldsymbol{\eta}_l(t) \;+\; \mathbf{g}(t)\,,\\
\boldsymbol{\eta}_l'(t) \quad &= \quad -\omega_{\boldsymbol{\sigma},l} C\big(\mu_{M,l}\,,\,\kappa_{M,l}\big)\,\varepsilon\big(\mathbf{v}(t)\big) \;-\; \omega_{\boldsymbol{\sigma},l}\boldsymbol{\eta}_l(t)\,, \qquad l = 1,\ldots,L\,,
\end{aligned} \tag{5.51}$$

$$\mathbf{v}(0) \;=\; \mathbf{v}^{(0)}\,, \qquad \boldsymbol{\sigma}(0) \;=\; \boldsymbol{\sigma}^{(0)}\,, \qquad \boldsymbol{\eta}(0) \;=\; \boldsymbol{\eta}^{(0)}\,,$$
$$\mathbf{v}(t)\big|_{\partial D_D} \;=\; \mathbf{0}\,, \qquad \mathbf{n}^\top\boldsymbol{\sigma}(t)\big|_{\partial D_N} \;=\; \mathbf{0}\,,$$

$t \in [0, t_1]$, *from* (5.23), *has a unique solution*

$$(\mathbf{v}, \boldsymbol{\sigma}, \boldsymbol{\eta})^\top \quad \in \quad C^1\big([0,t_1], X\big) \cap C\big([0,t_1], \mathcal{D}(A)\big)\,. \tag{5.52}$$

Proof. According to Theorem 72, the initial-boundary value problem (5.34) has a unique solution $(\mathbf{v}, \boldsymbol{\sigma}_H, \boldsymbol{\sigma}_M)^\top \in C^1\big([0,t_1], X\big) \cap C\big([0,t_1], \mathcal{D}(B)\big)$. By Theorem 65, this is equivalent to

$$\begin{aligned}
(\mathbf{v}, \boldsymbol{\sigma}, \boldsymbol{\eta})^\top \quad &:= \quad \tilde{T}^{-1}(\mathbf{v}, \boldsymbol{\sigma}_H, \boldsymbol{\sigma}_M)^\top\\
&= \quad \Big(\mathbf{v}\,,\; \boldsymbol{\sigma}_H + \sum_{l=1}^{L}\boldsymbol{\sigma}_{M,l}\,,\; -\omega_{\boldsymbol{\sigma},1}\boldsymbol{\sigma}_{M,1}\,,\; \ldots\,,\; -\omega_{\boldsymbol{\sigma},L}\boldsymbol{\sigma}_{M,L}\Big)^\top
\end{aligned}$$

being the unique solution of (5.51) with the property (5.52).

Alternatively, we could also prove this result directly, using Corollary 18 in section 3.2 together with Proposition 75. □

Theorem 78. *The energy of the solution* $u := (\mathbf{v}, \boldsymbol{\sigma}, \boldsymbol{\eta})^\top$ *of the homogeneous version of* (5.51) *with* $\tilde{\mathbf{f}} = \mathbf{0}$ *and* $\mathbf{g} = \mathbf{0}$ *has the time derivative*

$$\begin{aligned}
&\frac{d}{dt}\,\frac{1}{2}\|u(t)\|_T^2\\
&\quad = \quad -\int_D \sum_{l=1}^{L} \frac{1}{\omega_{\boldsymbol{\sigma},l}(x)} C\Big(\frac{1}{\mu_{M,l}(x)}\,,\,\frac{1}{\kappa_{M,l}(x)}\Big)\boldsymbol{\eta}_l(x,t) : \boldsymbol{\eta}_l(x,t)\;dx\\
&\quad \leq \quad 0\,,
\end{aligned} \tag{5.53}$$

$t \in [0, t_1]$. *In particular it holds*

$$\|u(t)\|_T \quad \leq \quad \|u_0\|_T \,, \qquad\qquad t \in [0, t_1] \,,$$

where $u_0 := \big(\mathbf{v}^{(0)}, \boldsymbol{\sigma}^{(0)}, \boldsymbol{\eta}^{(0)}\big)^\top$.

Proof. For $w := (\mathbf{v}, \boldsymbol{\sigma}_H, \boldsymbol{\sigma}_M)^\top := \tilde{T}u$, from Lemma 25 in section 4.1 it follows $\frac{d}{dt}\frac{1}{2}\|u(t)\|_T^2 = \frac{d}{dt}\frac{1}{2}\|w(t)\|_E^2$, $t \in [0, t_1]$, and the right-hand side of this equation has the explicit form (5.50). So we can simply use the definition (5.24) of T and plug the explicit form of $w = \tilde{T}u$ into that term. □

Remark 79. *Again, equation* (5.53) *allows the interpretation of the term*

$$\sum_{l=1}^{L} \frac{1}{\omega_{\boldsymbol{\sigma},l}} C\Big(\frac{1}{\mu_{M,l}}\,,\,\frac{1}{\kappa_{M,l}}\Big)\boldsymbol{\eta}_l : \boldsymbol{\eta}_l$$

to be the spatial density of the energy dissipation rate of a solution $u = (\mathbf{v}, \boldsymbol{\sigma}, \boldsymbol{\eta})^\top$ *of the homogeneous version of* (5.51) *with* $\tilde{\mathbf{f}} = \mathbf{0}$ *and* $\mathbf{g} = \mathbf{0}$. □

Corollary 80. *Let* $u_0 = \big(\mathbf{v}^{(0)}, \boldsymbol{\sigma}^{(0)}, \boldsymbol{\eta}^{(0)}\big)^\top \in \mathcal{D}(A)$, *and let* $u = (\mathbf{v}, \boldsymbol{\sigma}, \boldsymbol{\eta})^\top \in C^1\big([0, t_1], X\big) \cap C\big([0, t_1], \mathcal{D}(A)\big)$ *be the unique solution of the homogeneous version of our original initial-boundary value problem* (5.51) *with* $\tilde{\mathbf{f}} = \mathbf{0}$ *and* $\mathbf{g} = \mathbf{0}$. *Then there is* $M \geq 1$, *such that*

$$\|u(t)\|_X \;\leq\; M\, \|u_0\|_X \,, \qquad\qquad t \in [0, t_1] \,. \tag{5.54}$$

Proof. This directly follows from Theorem 78 and Lemma 63, since

$$\|u(t)\|_X \quad \lesssim \quad \|u(t)\|_T \quad \leq \quad \|u_0\|_T \quad \lesssim \quad \|u_0\|_X \,,$$

$t \in [0, t_1]$. □

Theorem 81. *The constant* M *in* (5.54) *cannot be chosen equal to 1.*

Proof. Let u be the unique solution of the homogeneous version of (5.51) with $\tilde{\mathbf{f}} = \mathbf{0}$ and

$$u(0) \quad = \quad u_0 \quad = \quad \begin{pmatrix} \mathbf{v}^{(0)} \\ \boldsymbol{\sigma}^{(0)} \\ \boldsymbol{\eta}_1^{(0)} \\ \vdots \\ \boldsymbol{\eta}_L^{(0)} \end{pmatrix} \quad := \quad \begin{pmatrix} 0 \\ 2\Big(\sum_{l=1}^{L} \|\omega_{\boldsymbol{\sigma},l}\|_{L^\infty(D)}\Big)\Phi \\ \Phi \\ \vdots \\ \Phi \end{pmatrix} \,,$$

where $0 \neq \Phi \in S \subseteq L^2(D, \mathbb{R}^{3\times 3}_{\mathrm{sym}})$. Then $u_0 \in \mathcal{D}(A)$, and

$$Au_0 \quad = \quad \Big(-2\,\big(\sum_{l=1}^{L} \|\omega_{\boldsymbol{\sigma},l}\|_{L^\infty(D)}\big)\vartheta \operatorname{div}\Phi\,,\ -L\Phi\,,\ \omega_{\boldsymbol{\sigma},1}\Phi\,,\ \ldots\,,\ \omega_{\boldsymbol{\sigma},L}\Phi\Big)^\top$$

and

$$\begin{aligned}
\frac{d}{dt}\|u(t)\|_X^2\Big|_{t=0} \quad &= \quad \frac{d}{dt}\big(u(t),u(t)\big)_X\Big|_{t=0} \quad = \quad 2\,\big(u'(0),u(0)\big)_X \\
&= \quad -2\,\big(Au(0),u(0)\big)_X \quad = \quad -2\,(Au_0,u_0)_X \\
&= \quad 2\int_D 2L\Big(\sum_{l=1}^{L}\|\omega_{\boldsymbol{\sigma},l}\|_{L^\infty(D)}\Big)\Phi:\Phi - \sum_{l=1}^{L}\omega_{\boldsymbol{\sigma},l}\Phi:\Phi\;dx \\
&\geq \quad 2\int_D 2L\Big(\sum_{l=1}^{L}\|\omega_{\boldsymbol{\sigma},l}\|_{L^\infty(D)}\Big)\Phi:\Phi - \sum_{l=1}^{L}\|\omega_{\boldsymbol{\sigma},l}\|_{L^\infty(D)}\Phi:\Phi\;dx \\
&= \quad 2\int_D (2L-1)\Big(\sum_{l=1}^{L}\|\omega_{\boldsymbol{\sigma},l}\|_{L^\infty(D)}\Big)\Phi:\Phi\;dx \\
&> \quad 0\,,
\end{aligned}$$

as $L \geq 1$. Hence there is $t \in (0,t_1]$ with $\|u(t)\|_X > \|u(0)\|_X = \|u_0\|_X$. $\square$

Chapter 6

The Parameter-to-Solution-Map

6.1 The Abstract Case

Throughout this section let $\big(X, (.\,,.)_X\big)$ be a real Hilbert space, $\|.\|_X$ the norm induced by $(.\,,.)_X$ and $t_1 > 0$.

The following assumption is motivated by the properties of the operator B from (5.29). Of particular interest is its decomposition $B = -P_1 Q + P_2$ derived in Lemma 59.

Assumption 82. In the sequel let $\emptyset \neq U \subseteq \mathcal{L}\big(X, \|.\|_X\big)^2$ denote a set of pairs $P = (P_1, P_2)$ of bounded linear operators $P_1, P_2 \in \mathcal{L}\big(X, \|.\|_X\big)$ and $Q : X \supseteq \mathcal{D}(Q) \to X$ a not necessarily bounded linear operator with domain of definition $\mathcal{D}(Q)$, such that the following holds true:

- For $P = (P_1, P_2) \in U$, the operator P_1 is self-adjoint, monotone and boundedly invertible with respect to the scalar product $(.\,,.)_X$.
- With $\|.\|_{\mathcal{L}(X,\|.\|_X)}$ denoting the operator norm on $\mathcal{L}\big(X, \|.\|_X\big)$, there exists a normed subspace $\big(\mathcal{T}U, \|.\|_{\mathcal{T}U,X}\big)$ of $\mathcal{L}\big(X, \|.\|_X\big)^2$ equipped with the norm
$$\|P\|_{\mathcal{T}U,X} \quad := \quad \max\Big\{\|P_1\|_{\mathcal{L}(X,\|.\|_X)}\,,\ \|P_2\|_{\mathcal{L}(X,\|.\|_X)}\Big\}\,, \tag{6.1}$$
$P = (P_1, P_2) \in \mathcal{T}U$, such that $U \subseteq \mathcal{T}U$ and U is open in $\big(\mathcal{T}U, \|.\|_{\mathcal{T}U,X}\big)$.
- For $P = (P_1, P_2) \in U$, the operator
$$\beta(P) := -P_1 Q + P_2 : \quad X \supseteq \mathcal{D}(Q) \quad \to \quad X$$
with domain of definition $\mathcal{D}\big(\beta(P)\big) := \mathcal{D}(Q)$ is maximal monotone in the sense of Definition 1 with respect to the scalar product
$$(w_1\,, w_2)_{E,P} \quad := \quad (P_1^{-1} w_1, w_2)_X\,, \qquad\qquad w_1, w_2 \in X\,,$$

which by Lemma 36 is well-defined.

The additional index E in the notation $(.\,,.)_{E,P}$ is added, to indicate that the scalar product $(.\,,.)_{E,P}$ plays the role of the scalar product $(.\,,.)_E$ in section 4.1. By $\|.\|_{E,P}$ we denote the norm induced by $(.\,,.)_{E,P}$. The graph norm of Q with respect to $\|.\|_X$ is denoted by

$$\|w\|_{Q,X} \quad := \quad \|Qw\|_X + \|w\|_X\,, \qquad\qquad w \in \mathcal{D}(Q)\,.$$

The graph norm of an operator $\beta(P)$ corresponding to parameters $P \in U$ with respect to the norm $\|.\|_{E,\tilde{P}}$ corresponding to parameters $\tilde{P} \in U$ is denoted by

$$\|w\|_{\beta(P),E,\tilde{P}} \quad := \quad \|\beta(P)w\|_{E,\tilde{P}} + \|w\|_{E,\tilde{P}}\,, \qquad\qquad w \in \mathcal{D}(Q)\,.$$

The space $\mathcal{T}U$ can also be equipped with the norm

$$\|P\|_{\mathcal{T}U,E,\tilde{P}} \quad := \quad \max\Big\{\|P_1\|_{\mathcal{L}(X,\|.\|_{E,\tilde{P}})}\,,\ \|P_2\|_{\mathcal{L}(X,\|.\|_{E,\tilde{P}})}\Big\}\,, \qquad\qquad P \in \mathcal{T}U\,,$$

for any fixed $\tilde{P} \in U$.

Although U is no vector space, we still use the notation $\big(U, \|.\|_{\mathcal{T}U,X}\big)$, etc. to indicate that we consider the restriction of the metric of $\big(\mathcal{T}U, \|.\|_{\mathcal{T}U,X}\big)$ onto U. $\square$

The next lemma is an adaptation of Lemma 36.

Lemma 83. *For any $P \in U$, we have*

$$\frac{1}{\sqrt{\|P_1\|_{\mathcal{L}(X,\|.\|_X)}}}\,\|w\|_X \quad \leq \quad \|w\|_{E,P} \quad \leq \quad \sqrt{\|P_1^{-1}\|_{\mathcal{L}(X,\|.\|_X)}}\,\|w\|_X\,,$$

$w \in X$. It also holds

$$k_P\|w\|_{Q,X} \quad \leq \quad \|w\|_{\beta(P),E,P} \quad \leq \quad K_P\|w\|_{Q,X}\,, \tag{6.2}$$

$w \in \mathcal{D}(Q)$, with

$$\begin{aligned} k_P \quad &:= \quad \Bigg(\sqrt{\|P_1\|_{\mathcal{L}(X,\|.\|_X)}} \\ &\qquad\qquad \max\Big\{\|P_1^{-1}\|_{\mathcal{L}(X,\|.\|_X)}\,,\ \|P_1^{-1}\|_{\mathcal{L}(X,\|.\|_X)}\|P_2\|_{\mathcal{L}(X,\|.\|_X)} + 1\Big\}\Bigg)^{-1}\,, \\ K_P \quad &:= \quad \sqrt{\|P_1^{-1}\|_{\mathcal{L}(X,\|.\|_X)}}\ \max\Big\{\|P_1\|_{\mathcal{L}(X,\|.\|_X)}\,,\ \|P_2\|_{\mathcal{L}(X,\|.\|_X)} + 1\Big\}\,. \end{aligned}$$

Furthermore, for fixed $\hat{P} \in U$ there is a neighborhood $\Omega \subseteq U$ of $\hat{P}$ and a constant $c > 0$, such that

$$\frac{1}{c}\,\|w\|_{Q,X} \quad \leq \quad \|w\|_{\beta(P),E,P} \quad \leq \quad c\,\|w\|_{Q,X}\,, \qquad P \in \Omega\,,\ w \in \mathcal{D}(Q)\,. \tag{6.3}$$

Proof. The first two estimates are applications of Lemma 36 with $\tilde{P} = P$. We prove (6.3).

Let $\hat{P} \in U$. Since U is open in $\big(\mathcal{T}U, \|.\|_{\mathcal{T}U}\big)$, there is an open ball $B(\hat{P}, \delta_1) \subseteq \mathcal{L}\big(X, \|.\|_X\big)^2$ round $\hat{P}$ with radus $\delta_1 > 0$, such that $B(\hat{P}, \delta_1) \cap \mathcal{T}U \subseteq U$. Since the map $B(\hat{P}, \delta_1) \cap \mathcal{T}U \to \mathcal{L}(X, \|.\|_X)$, $(P_1, P_2) \mapsto P_1^{-1}$ is continuous, there is $\delta > 0$ with $\delta_1 \geq \delta$, such that $P_1^{-1} \in B(\hat{P}_1^{-1}, \delta_1)$ for every $(P_1, P_2) \in B(\hat{P}, \delta) \cap \mathcal{T}U$. So if we define $\Omega := B(\hat{P}, \delta) \cap \mathcal{T}U$ and

$$d \;:=\; \max\big\{\|\hat{P}_1\|_{\mathcal{L}(X,\|.\|_X)},\, \|\hat{P}_1^{-1}\|_{\mathcal{L}(X,\|.\|_X)},\, \|\hat{P}_2\|_{\mathcal{L}(X,\|.\|_X)}\big\} + \delta_1\,,$$

we get

$$\|P_1\|_{\mathcal{L}(X,\|.\|_X)}\,,\; \|P_1^{-1}\|_{\mathcal{L}(X,\|.\|_X)}\,,\; \|P_2\|_{\mathcal{L}(X,\|.\|_X)} \quad\leq\quad d$$

for $(P_1, P_2) \in \Omega$. Now, from the last part of Lemma 36, estimate (6.3) follows. $\square$

Throughout this section we make use of the equivalence of these norms and switch between them whenever one norm seems more useful or more natural than the other.

Lemma 84. *For $P \in U$ it is*

$$\begin{aligned} &C^1\Big([0,t_1], \big(X, \|.\|_{E,P}\big)\Big) \cap C\bigg([0,t_1], \Big(\mathcal{D}\big(\beta(P)\big), \|.\|_{\beta(P),E,P}\Big)\bigg) \\ &\qquad = \quad C^1\Big([0,t_1], (X, \|.\|_X)\Big) \cap C\Big([0,t_1], \big(\mathcal{D}(Q), \|.\|_{Q,X}\big)\Big) \end{aligned}$$

and

$$W^{k,1}\big((0,t_1), (X, \|.\|_X)\big) \quad=\quad W^{k,1}\big((0,t_1), (X, \|.\|_{E,P})\big)\,,$$

$k \in \mathbb{N}_0$. Therefore we can drop the norm symbols in the notation of these spaces.

Proof. By definition, it is $\mathcal{D}\big(\beta(P)\big) = \mathcal{D}(Q)$. So this statement is a consequence of the norm equivalences proven in Lemma 83. $\square$

Definition 85. *The* ***abstract parameter-to-solution-map*** *is defined as*

$$\begin{aligned} G \,:\;\; &\big(U, \|.\|_{TU,X}\big) \,\times\, \big(\mathcal{D}(Q), \|.\|_{Q,X}\big) \,\times\, W^{1,1}\big((0,t_1), (X, \|.\|_X)\big) \\ &\qquad\to\quad C^1\Big([0,t_1], (X, \|.\|_X)\Big) \cap C\Big([0,t_1], \big(\mathcal{D}(Q), \|.\|_{Q,X}\big)\Big)\,, \\ &(P, w_0, f) \quad\mapsto\quad w\,, \end{aligned} \tag{6.4}$$

where w is the classical solution of the inhomogeneous evolution equation

$$\begin{aligned} w'(t) \,+\, \beta(P)\,w(t) \quad&=\quad f(t)\,, \qquad\qquad t \in [0,t_1]\,, \\ w(0) \quad&=\quad w_0\,. \end{aligned} \tag{6.5}$$

$\square$

Notation 86. In a canonical way, we regard the vector space $\big(\mathcal{T}U, \|.\|_{\mathcal{T}U,X}\big) \times \big(\mathcal{D}(Q), \|.\|_{Q,X}\big) \times W^{1,1}\big((0,t_1),(X,\|.\|_X)\big)$, which contains the domain of G as a subset, to be endowed with the norm

$$\big\|(P, w_0, f)\big\|_1 \quad = \quad \|P\|_{\mathcal{T}U,X} \;+\; \|w_0\|_{Q,X} \;+\; \|f\|_{W^{1,1}((0,t_1),(X,\|.\|_X))} \tag{6.6}$$

for every (P, w_0, f). $\square$

6.1.1 Fréchet-Differentiability

In this section we will prove the Fréchet-differentiability of G considered as a mapping into the bigger space $C\big([0,t_1],(X,\|.\|_X)\big)$. The methods we use to accomplish this are slight adaptations of the ideas developed in [14].

Lemma 87. *In this lemma we consider the map G defined in* (6.4) *with the bigger codomain $C\big([0,t_1],(\mathcal{D}(Q),\|.\|_{Q,X})\big)$.*

For every $P \in U$, the map

$$\begin{aligned} G(P,\bullet,\bullet):\quad \big(\mathcal{D}(Q),\|.\|_{Q,X}\big) \times W^{1,1}\big((0,t_1),(X,\|.\|_X)\big) & \\ \to \quad & C\Big([0,t_1],\big(\mathcal{D}(Q),\|.\|_{Q,X}\big)\Big), \\ (w_0,f) \quad \mapsto \quad & G(P,w_0,f) \end{aligned}$$

is linear.

As usual, the space $\big(\mathcal{D}(Q),\|.\|_{Q,X}\big) \times W^{1,1}\big((0,t_1),(X,\|.\|_X)\big)$ is assumed to be equipped with any norm which is equivalent to

$$\|(w_0,f)\|_1 \quad = \quad \|w_0\|_{Q,X} \;+\; \|f\|_{W^{1,1}((0,t_1),(X,\|.\|_X))}$$

for each (w_0,f). For any $c>0$ there is $\gamma_c>0$, such that for every element P of the set

$$\Big\{\; P \in U : \quad \|P\|_{\mathcal{T}U,X}\,,\; \|P_1^{-1}\|_{\mathcal{L}(X,\|.\|_X)} \quad \le \quad c \;\Big\}, \tag{6.7}$$

$G(P,\bullet,\bullet)$ is bounded by γ_c.

Proof. Let $P=(P_1,P_2)\in U$, $w_0,w_1 \in \mathcal{D}(Q)$, $f,g \in W^{1,1}\big((0,t_1),X\big)$, and $\alpha \in \mathbb{R}$. By definition of G, it is

$$\begin{aligned} G(P,w_0,f)'(t) \;+\; \beta(P)G(P,w_0,f)(t) \quad &= \quad f(t)\,, \qquad && t \in [0,t_1]\,, \\ G(P,w_0,f)(0) \quad &= \quad w_0\,, \end{aligned}$$

and

$$\begin{aligned} G(P,w_1,g)'(t) \;+\; \beta(P)G(P,w_1,g)(t) \quad &= \quad g(t)\,, \qquad && t \in [0,t_1]\,, \\ G(P,w_1,g)(0) \quad &= \quad w_1\,. \end{aligned}$$

So by multiplication of the second evolution equation by α and subsequent addition of both equations, we find that $w := G(P, w_0, f) + \alpha G(P, w_1, g)$ classically solves

$$\begin{aligned} w'(t) + \beta(P)w(t) &= (f + \alpha g)(t), \qquad t \in [0, t_1], \\ w(0) &= w_0 + \alpha w_1 . \end{aligned}$$

Since this solution is unique, it follows again from the definition of G, that $w = G(P, w_0 + \alpha w_1, f + \alpha g)$. Hence, $G(P, \bullet, \bullet)$ is linear.

To prove the second statement, let $c > 0$, P be an element of the set (6.7), and $(w_0, f) \in \mathcal{D}(Q) \times W^{1,1}\big((0, t_1), X\big)$. Since by Lemma 15, the classical solution of (6.5) coincides with the mild solution, it is

$$G(P, w_0, f)(t) = S_{\beta(P)}(t)w_0 + \int_0^t S_{\beta(P)}(t - s)f(s)\, ds ,$$

$t \in [0, t_1]$. And according to Lemma 83 and estimate (3.19),

$$\begin{aligned} &\big\|G(P, w_0, f)\big\|_{C([0,t_1],(\mathcal{D}(Q),\|\cdot\|_{Q,X}))} \\ &\leq \frac{1}{k_P} \big\|G(P, w_0, f)\big\|_{C([0,t_1],(\mathcal{D}(Q),\|\cdot\|_{\beta(P),E,P}))} \\ &\leq \frac{1}{k_P}\Big(\|w_0\|_{\beta(P),E,P} + (2c_{CW} + 1)\, \|f\|_{W^{1,1}((0,t_1),(X,\|\cdot\|_{E,P}))}\Big) \\ &\leq \frac{1}{k_P}\Big(K_P\|w_0\|_{Q,X} + (2c_{CW} + 1)\sqrt{\|P_1^{-1}\|_{\mathcal{L}(X,\|\cdot\|_X)}}\, \|f\|_{W^{1,1}((0,t_1),(X,\|\cdot\|_X))}\Big) \\ &\leq \frac{1}{k_P} \max\Big\{K_P\, , \, (2c_{CW} + 1)\sqrt{\|P_1^{-1}\|_{\mathcal{L}(X,\|\cdot\|_X)}}\Big\}\, \|(w_0, f)\|_1 \\ &\leq \sqrt{c}\, \max\{c, c^2 + 1\} \max\Big\{\sqrt{c} \max\{c, c + 1\}\, , \, (2c_{CW} + 1)\sqrt{c}\Big\}\, \|(w_0, f)\|_1 . \end{aligned}$$

□

Lemma 88. *For any fixed $\hat{P} \in U$, the space*

$$\Big\{(w_0, f) \in \mathcal{D}(Q) \times W^{2,1}\big((0, t_1), X\big) : \quad \beta(\hat{P})w_0 - f(0) \in \mathcal{D}(Q)\Big\} \tag{6.8}$$

is dense in

$$\big(\mathcal{D}(Q), \|.\|_{Q,X}\big) \times W^{1,1}\big((0, t_1), (X, \|.\|_X)\big) .$$

Proof. We adapt the proof given in [14].

For $\hat{P} \in U$, the space $\mathcal{D}\big(\beta(\hat{P})^2\big) := \big\{w_0 \in \mathcal{D}(Q) : \beta(\hat{P})w_0 \in \mathcal{D}(Q)\big\}$ is dense in $\big(\mathcal{D}(Q), \|.\|_{\beta(\hat{P}),E,\hat{P}}\big)$. This is the statement of Lemma 7.2 in [5]. Because of (6.2), it is also dense in $\big(\mathcal{D}(Q), \|.\|_{Q,X}\big)$. Furthermore, by Lemma 2(a) in section

3.1, $\mathcal{D}(Q)$ is dense in $\big(X, \|.\|_X\big)$. And as a consequence of Lemma A.4, the space $W^{2,1}\big((0,t_1),X\big)$ is dense in $W^{1,1}\big((0,t_1),(X,\|.\|_X)\big)$. In the sequel we also make use of Lemma A.2.

Now let $(w_0, f) \in \mathcal{D}(Q) \times W^{1,1}\big((0,t_1),X\big)$ be arbitrary. Then there are $w_{0,n} \in \mathcal{D}\big(\beta(\hat{P})^2\big)$, $z_n \in \mathcal{D}(Q)$ and $g_n \in W^{2,1}\big((0,t_1),X\big)$, $n \in \mathbb{N}$, with $\|w_{0,n} - w_0\|_{Q,X} \to 0$, $\|z_n - f(0)\|_X \to 0$ and $\|g_n - f\|_{W^{1,1}((0,t_1),(X,\|.\|_X))} \to 0$ for $n \to \infty$. Define $f_n(t) \coloneqq g_n(t) + \big(z_n - g_n(0)\big)$, $t \in (0,t_1)$, $n \in \mathbb{N}$. Then $f_n \in W^{2,1}\big((0,t_1),X\big)$, $n \in \mathbb{N}$, and $\|f_n - f\|_{W^{1,1}((0,t_1),(X,\|.\|_X))} \to 0$, $n \to \infty$. Furthermore, $f_n(0) = z_n \in \mathcal{D}(Q)$ and $\beta(\hat{P})w_{0,n} \in \mathcal{D}(Q)$. So also $\beta(\hat{P})w_{0,n} - f_n(0) \in \mathcal{D}(Q)$. This means, $(w_{0,n}, f_n)$ is an element of the space (6.8), $n \in \mathbb{N}$, and $(w_{0,n}, f_n) \to (w_0, f)$ in $\big(\mathcal{D}(Q), \|.\|_{Q,X}\big) \times W^{1,1}\big((0,t_1),(X,\|.\|_X)\big)$ for $n \to \infty$. □

Proposition 89. *The map G from* (6.4) *considered with the bigger codomain,*

$$G:\ \big(U, \|.\|_{\mathcal{T}U,X}\big) \times \big(\mathcal{D}(Q), \|.\|_{Q,X}\big) \times W^{1,1}\big((0,t_1),(X,\|.\|_X)\big) \\ \to \quad C\big([0,t_1]\,,\,(\mathcal{D}(Q),\|.\|_{Q,X})\big)\,,$$

is continuous.

Proof. In a first step, let $\hat{\Pi} = (\hat{P}, \hat{w}_0, \hat{f}) \in U \times \mathcal{D}(Q) \times W^{1,1}\big((0,t_1),X\big)$, such that $(\hat{w}_0, \hat{f})$ is an element of the space defined in (6.8). For $\Pi = (P, w_0, f) \in \mathcal{T}U \times \mathcal{D}(Q) \times W^{1,1}\big((0,t_1),X\big)$ with $\hat{P} + P \in U$, we have

$$\begin{aligned} \frac{d}{dt}G(\hat{\Pi}+\Pi)(t) \ + \ \beta(\hat{P}+P)\,G(\hat{\Pi}+\Pi)(t) \quad &= \quad \hat{f}(t) \ + \ f(t)\,, \\ G(\hat{\Pi}+\Pi)(0) \quad &= \quad \hat{w}_0 \ + \ w_0 \end{aligned} \tag{6.9}$$

and

$$\begin{aligned} \frac{d}{dt}G(\hat{\Pi})(t) \ + \ \beta(\hat{P})\,G(\hat{\Pi})(t) \quad &= \quad \hat{f}(t)\,, \\ G(\hat{\Pi})(0) \quad &= \quad \hat{w}_0\,, \end{aligned} \tag{6.10}$$

$t \in [0,t_1]$. Thus for all $t \in [0,t_1]$:

$$\begin{aligned} \frac{d}{dt}\big[G(\hat{\Pi}+\Pi) - G(\hat{\Pi})\big](t) \ + \ \beta(\hat{P}+P)\,\big[G(\hat{\Pi}+\Pi) - G(\hat{\Pi})\big](t) & \\ = \quad f(t) \ - \ \beta(P)\,G(\hat{\Pi})(t)\,, & \\ \big[G(\hat{\Pi}+\Pi) - G(\hat{\Pi})\big](0) \quad = \quad w_0\,. & \end{aligned} \tag{6.11}$$

Along with $G(\hat{\Pi}+\Pi)$ and $G(\hat{\Pi})$ being the classical solutions of (6.9) and (6.10), respectively, also $G(\hat{\Pi}+\Pi) - G(\hat{\Pi})$ is the classical solution of (6.11). So according to Lemma 15, this function is also a mild solution and has the form

$$\begin{aligned}
&\big[G(\hat{\Pi}+\Pi)-G(\hat{\Pi})\big](t)\\
&\qquad = \quad S_{\beta(\hat{P}+P)}(t)\,w_0 \;+\; \int_0^t S_{\beta(\hat{P}+P)}(t-s)\,\big[f(s)-\beta(P)G(\hat{\Pi})(s)\big]\,ds\,,
\end{aligned}$$

$t \in [0,t_1]$. As $(\hat{w}_0,\hat{f})$ is an element of the space (6.8), Lemma 20 states that

$$G(\hat{\Pi}) \quad\in\quad C^1\Big([0,t_1],\big(\mathcal{D}(Q),\|.\|_{\beta(\hat{P}),E,\hat{P}}\big)\Big) \quad=\quad C^1\Big([0,t_1],\big(\mathcal{D}(Q),\|.\|_{Q,X}\big)\Big)\,.$$

So

$$\beta(P)G(\hat{\Pi}) \quad=\quad -P_1QG(\hat{\Pi}) \,+\, P_2G(\hat{\Pi}) \quad\in\quad C^1\big([0,t_1],(X,\|.\|_X)\big)$$

and

$$f \,-\, \beta(P)G(\hat{\Pi}) \quad\in\quad W^{1,1}\big((0,t_1),(X,\|.\|_X)\big)\,.$$

Now Lemma 83 and Lemma 17 allow us to make the following estimate:

$$\begin{aligned}
&\Big\|\big[G(\hat{\Pi}+\Pi)-G(\hat{\Pi})\big](t)\Big\|_{Q,X}\\
&\qquad\le\quad c_1(P)\,\Big\|\big[G(\hat{\Pi}+\Pi)-G(\hat{\Pi})\big](t)\Big\|_{\beta(\hat{P}+P),E,\hat{P}+P}\\
&\qquad\le\quad c_2(P)\,\Big(\|w_0\|_{\beta(\hat{P}+P),E,\hat{P}+P} \,+\, \big\|f-\beta(P)\,G(\hat{\Pi})\big\|_{W^{1,1}((0,t_1),(X,\|.\|_{E,\hat{P}+P}))}\Big)\\
&\qquad\le\quad c_3(P)\,\Big(\|w_0\|_{Q,X} \,+\, \big\|f-\beta(P)\,G(\hat{\Pi})\big\|_{W^{1,1}((0,t_1),(X,\|.\|_X))}\Big)\\
&\qquad=\quad c_3(P)\,\Big(\|w_0\|_{Q,X} \,+\, \big\|f-(-P_1Q+P_2)\,G(\hat{\Pi})\big\|_{W^{1,1}((0,t_1),(X,\|.\|_X))}\Big)\\
&\qquad\le\quad c_3(P)\,\Big(\|w_0\|_{Q,X} \,+\, \|f\|_{W^{1,1}((0,t_1),(X,\|.\|_X))}\\
&\qquad\qquad\qquad\qquad +\, \|P\|_{\mathcal{T}U,X}\,\big\|G(\hat{\Pi})\big\|_{W^{1,1}((0,t_1),(\mathcal{D}(Q),\|.\|_{Q,X}))}\Big)
\end{aligned}$$

for each $t \in [0,t_1]$, with

$$\begin{aligned}
c_1(P) \quad&:=\quad \frac{1}{k_{\hat{P}+P}}\,,\\
c_2(P) \quad&:=\quad c_1(P)\,(2c_{CW}+1)\,,\\
c_3(P) \quad&:=\quad c_2(P)\,\max\Big\{K_{\hat{P}+P}\,,\;\sqrt{\|(\hat{P}_1+P_1)^{-1}\|_{\mathcal{L}(X,\|.\|_X)}}\Big\}\\
&=\quad c_2(P)\,K_{\hat{P}+P}\,.
\end{aligned}$$

So

$$
\begin{aligned}
&\big\|G(\hat{\Pi}+\Pi)-G(\hat{\Pi})\big\|_{C([0,t_1],(\mathcal{D}(Q),\|\cdot\|_{Q,X}))} \\
&\qquad\qquad \leq \quad c_3(P)\,\Big(\|w_0\|_{Q,X} \;+\; \|f\|_{W^{1,1}((0,t_1),(\mathcal{D}(Q),\|\cdot\|_{Q,X}))} \\
&\qquad\qquad\qquad\qquad + \;\|P\|_{\mathcal{T}U,X}\,\big\|G(\hat{\Pi})\big\|_{W^{1,1}((0,t_1),(\mathcal{D}(Q),\|\cdot\|_{Q,X}))}\Big) \\
&\qquad\qquad \longrightarrow\; 0\,, \qquad\qquad \Pi=(P,w_0,f)\;\to\;0\,.
\end{aligned}
$$

In a second step, let $(\hat{P},\hat{w}_0,\hat{f})$ be arbitrary in $U\times\mathcal{D}(Q)\times W^{1,1}\big((0,t_1),X\big)$ and $\varepsilon>0$. For P having the properties

$$
P\;\in\;\mathcal{T}U\,,\qquad \hat{P}+P\;\in\;U\,,\qquad \|P\|_{\mathcal{T}U,X}\;<\;\frac{1}{2\,\|\hat{P}_1^{-1}\|_{\mathcal{L}(X,\|\cdot\|_X)}}\;=:\;\delta_1\,, \tag{6.12}
$$

it is

$$
\|\hat{P}+P\|_{\mathcal{T}U,X}\,,\;\|(\hat{P}_1+P_1)^{-1}\|_{\mathcal{L}(X,\|\cdot\|_X)}\quad<\quad\max\Big\{\|\hat{P}\|_{\mathcal{T}U,X}+\delta_1\,,\;\frac{1}{\delta_1}\Big\}\quad=:\quad c\,.
$$

Indeed, the term $\|\hat{P}+P\|_{\mathcal{T}U,X}$ is simply estimated by using the triangle inequality, whereas for the term $\|(\hat{P}_1+P_1)^{-1}\|_{\mathcal{L}(X,\|\cdot\|_X)}$, this estimate can be achieved by applying the Neumann series in the following way.

$$
\begin{aligned}
\|(\hat{P}_1+P_1)^{-1}\|_{\mathcal{L}(X,\|\cdot\|_X)} \quad&\leq\quad \|\hat{P}_1^{-1}\|_{\mathcal{L}(X,\|\cdot\|_X)}\|(\mathrm{Id}+P_1\hat{P}_1^{-1})^{-1}\|_{\mathcal{L}(X,\|\cdot\|_X)} \\
&=\quad \|\hat{P}_1^{-1}\|_{\mathcal{L}(X,\|\cdot\|_X)}\Big\|\sum_{n=0}^{\infty}(-P_1\hat{P}_1^{-1})^n\Big\|_{\mathcal{L}(X,\|\cdot\|_X)} \\
&\leq\quad \|\hat{P}_1^{-1}\|_{\mathcal{L}(X,\|\cdot\|_X)}\sum_{n=0}^{\infty}\big(\|P_1\|_{\mathcal{L}(X,\|\cdot\|_X)}\|\hat{P}_1^{-1}\|_{\mathcal{L}(X,\|\cdot\|_X)}\big)^n \\
&=\quad \frac{\|\hat{P}_1^{-1}\|_{\mathcal{L}(X,\|\cdot\|_X)}}{1-\|P_1\|_{\mathcal{L}(X,\|\cdot\|_X)}\|\hat{P}_1^{-1}\|_{\mathcal{L}(X,\|\cdot\|_X)}} \\
&<\quad 2\,\|\hat{P}_1^{-1}\|_{\mathcal{L}(X,\|\cdot\|_X)} \\
&=\quad \frac{1}{\delta_1}\,,
\end{aligned}
$$

since $\|P_1\|<\delta_1$ by (6.12) and therefore also $\|P_1\|_{\mathcal{L}(X,\|\cdot\|_X)}\|\hat{P}_1^{-1}\|_{\mathcal{L}(X,\|\cdot\|_X)}<1/2<1$.

Due to Lemma 87, for all P satisfying (6.12) and all $(w_0,f),(w_1,g)\in\mathcal{D}(Q)\times$

$W^{1,1}\big((0,t_1),X\big)$ with $\|(w_0,f)\|_1, \|(w_1,g)\|_1 < \delta_2 \coloneqq \varepsilon/(6\gamma_c)$, it is

$$\begin{aligned}
&\big\|G(\hat{P}+P,\hat{w}_0+w_0,\hat{f}+f)-G(\hat{P}+P,\hat{w}_0+w_1,\hat{f}+g)\big\|_{C([0,t_1],(\mathcal{D}(Q),\|.\|_{Q,X}))}\\
&\quad= \big\|G(\hat{P}+P,w_0-w_1,f-g)\big\|_{C([0,t_1],(\mathcal{D}(Q),\|.\|_{Q,X}))}\\
&\quad\le \gamma_c\,\|(w_0-w_1,f-g)\|_1\\
&\quad\le \gamma_c\,\big(\|(w_0,f)\|_1+\|(w_1,g)\|_1\big)\\
&\quad< 2\gamma_c\delta_2\\
&\quad= \frac{\varepsilon}{3}\,.
\end{aligned}$$

Because of Lemma 88, there exists $(\hat{w}_1,\hat{g}) \in \mathcal{D}(Q)\times W^{1,1}\big((0,t_1),X\big)$ with $\|(\hat{w}_1,\hat{g})\|_1 < \delta_2$ such that $(\hat{w}_0+\hat{w}_1,\hat{f}+\hat{g})$ is an element of the space (6.8). And as we showed in the first step of this proof, there is $\delta_3>0$ such that $\|P\|_{\mathcal{T}U,X}<\delta_3$ implies

$$\big\|G(\hat{P}+P,\hat{w}_0+\hat{w}_1,\hat{f}+\hat{g})-G(\hat{P},\hat{w}_0+\hat{w}_1,\hat{f}+\hat{g})\big\|_{C([0,t_1],(\mathcal{D}(Q),\|.\|_{Q,X}))} < \frac{\varepsilon}{3}\,.$$

So for $(P,w_0,f)\in\mathcal{T}U\times\mathcal{D}(Q)\times W^{1,1}\big((0,t_1),X\big)$ fulfilling $\hat{P}+P\in U$, $\|P\|_{\mathcal{T}U,X}<\min\{\delta_1,\delta_3\}$ and $\|(w_0,f)\|_1<\delta_2$, it is

$$\begin{aligned}
&\big\|G(\hat{P}+P,\hat{w}_0+w_0,\hat{f}+f)-G(\hat{P},\hat{w}_0,\hat{f})\big\|_{C([0,t_1],(\mathcal{D}(Q),\|.\|_{Q,X}))}\\
&\quad\le \big\|G(\hat{P}+P,\hat{w}_0+w_0,\hat{f}+f)-G(\hat{P}+P,\hat{w}_0+\hat{w}_1,\hat{f}+\hat{g})\big\|_{C([0,t_1],(\mathcal{D}(Q),\|.\|_{Q,X}))}\\
&\qquad+ \big\|G(\hat{P}+P,\hat{w}_0+\hat{w}_1,\hat{f}+\hat{g})-G(\hat{P},\hat{w}_0+\hat{w}_1,\hat{f}+\hat{g})\big\|_{C([0,t_1],(\mathcal{D}(Q),\|.\|_{Q,X}))}\\
&\qquad+ \big\|G(\hat{P},\hat{w}_0+\hat{w}_1,\hat{f}+\hat{g})-G(\hat{P},\hat{w}_0,\hat{f})\big\|_{C([0,t_1],(\mathcal{D}(Q),\|.\|_{Q,X}))}\\
&\quad< \varepsilon\,,
\end{aligned}$$

which concludes the proof. □

Theorem 90. *The interpretation of G defined in* (6.4) *with the bigger codomain,*

$$\begin{aligned}
G:\big(U,\|.\|_{\mathcal{T}U,X}\big)\times\big(\mathcal{D}(Q),\|.\|_{Q,X}\big)\times W^{1,1}\big((0,t_1),(X,\|.\|_X)\big)&\\
\to C\big([0,t_1],(X,\|.\|_X)\big)\,,&
\end{aligned}\tag{6.13}$$

is Fréchet-differentiable, and the Fréchet-derivative

$$\begin{aligned}
DG:\ &\big(U,\|.\|_{\mathcal{T}U,X}\big)\times\big(\mathcal{D}(Q),\|.\|_{Q,X}\big)\times W^{1,1}\big((0,t_1),(X,\|.\|_X)\big)\\
&\to \mathcal{L}\Big(\big(\mathcal{T}U,\|.\|_{\mathcal{T}U,X}\big)\times\big(\mathcal{D}(Q),\|.\|_{Q,X}\big)\times W^{1,1}\big((0,t_1),(X,\|.\|_X)\big)\,,\\
&\qquad C\big([0,t_1],(X,\|.\|_X)\big)\Big)
\end{aligned}$$

is given by

$$\big[DG(\hat\Pi)\Pi\big](t) \;=\; S_{\beta(\hat P)}(t)\,w_0 + \int_0^t S_{\beta(\hat P)}(t-s)\,\big[f(s)-\beta(P)G(\hat\Pi)(s)\big]\,ds\,, \tag{6.14}$$

$t\in[0,t_1]$*, with the notation* $\hat\Pi=(\hat P,\hat w_0,\hat f)$ *and* $\Pi=(P,w_0,f)$*. This is the mild solution of*

$$\begin{aligned} w'(t) \;+\; \beta(\hat P)\,w(t) \;&=\; f(t) \;-\; \beta(P)\,G(\hat\Pi)(t)\,, \qquad t\in[0,t_1]\,,\\ w(0) \;&=\; w_0\,. \end{aligned}$$

Proof. Let $\hat\Pi=(\hat P,\hat w_0,\hat f)\in U\times\mathcal{D}(Q)\times W^{1,1}\big((0,t_1),X\big)$ and let $\Pi=(P,w_0,f)\in \mathcal{T}U\times\mathcal{D}(Q)\times W^{1,1}\big((0,t_1),X\big)$ such that $\hat P+P\in U$. Our aim is to estimate the norm of the function $G(\hat\Pi+\Pi)-G(\hat\Pi)-DG(\hat\Pi)\Pi$.

As $G(\hat\Pi+\Pi)$ and $G(\hat\Pi)$ classically solve (6.9) and (6.10), respectively, $G(\hat\Pi+\Pi)-G(\hat\Pi)$ is the classical solution of

$$\begin{aligned} \frac{d}{dt}\big[G(\hat\Pi+\Pi)-G(\hat\Pi)\big](t) \;+\; \beta(\hat P)\,\big[G(\hat\Pi+\Pi)-G(\hat\Pi)\big](t)&\\ \;=\;& f(t) \;-\; \beta(P)\,G(\hat\Pi+\Pi)(t)\,,\\ \big[G(\hat\Pi+\Pi)-G(\hat\Pi)\big](0) \;=\;& w_0\,, \end{aligned}$$

$t\in[0,t_1]$. By Lemma 15 it is also the mild solution of this equation. So we can write

$$\begin{aligned} &\big[G(\hat\Pi+\Pi)-G(\hat\Pi)\big](t)\\ &\quad=\; S_{\beta(\hat P)}(t)\,w_0 \;+\; \int_0^t S_{\beta(\hat P)}(t-s)\,\big[f(s)-\beta(P)G(\hat\Pi+\Pi)(s)\big]\,ds\,, \end{aligned} \tag{6.15}$$

$t\in[0,t_1]$. Then, subtracting (6.14) from (6.15), we get

$$\begin{aligned} &\big[G(\hat\Pi+\Pi)-G(\hat\Pi)-DG(\hat\Pi)\Pi\big](t)\\ &\qquad=\; -\int_0^t S_{\beta(\hat P)}(t-s)\,\beta(P)\big[G(\hat\Pi+\Pi)-G(\hat\Pi)\big](s)\,ds\,, \end{aligned}$$

$t\in[0,t_1]$. It follows

$$\begin{aligned} &\Big\|\big[G(\hat\Pi+\Pi)-G(\hat\Pi)-DG(\hat\Pi)\Pi\big](t)\Big\|_X\\ &\qquad\le\; c_1\,\Big\|\big[G(\hat\Pi+\Pi)-G(\hat\Pi)-DG(\hat\Pi)\Pi\big](t)\Big\|_{E,\hat P}\\ &\qquad\le\; c_1\int_0^t \Big\|S_{\beta(\hat P)}(t-s)\,\beta(P)\big[G(\hat\Pi+\Pi)-G(\hat\Pi)\big](s)\Big\|_{E,\hat P}\,ds \end{aligned}$$

$$
\begin{aligned}
&\leq \quad c_1 \int_0^t \left\| \beta(P)\big[G(\hat{\Pi}+\Pi) - G(\hat{\Pi})\big](s) \right\|_{E,\hat{P}} ds \\
&\leq \quad c_2 \int_0^t \left\| \beta(P)\big[G(\hat{\Pi}+\Pi) - G(\hat{\Pi})\big](s) \right\|_X ds \\
&= \quad c_2 \int_0^t \left\| (-P_1 Q + P_2)\big[G(\hat{\Pi}+\Pi) - G(\hat{\Pi})\big](s) \right\|_X ds \\
&\leq \quad c_2 \, \|P\|_{\mathcal{T}U,X} \int_0^t \left\| \big[G(\hat{\Pi}+\Pi) - G(\hat{\Pi})\big](s) \right\|_{Q,X} ds \\
&\leq \quad c_2 \, \|\Pi\|_1 \int_0^t \left\| \big[G(\hat{\Pi}+\Pi) - G(\hat{\Pi})\big](s) \right\|_{Q,X} ds \\
&\leq \quad t\, c_2 \, \|\Pi\|_1 \left\| G(\hat{\Pi}+\Pi) - G(\hat{\Pi}) \right\|_{C([0,t],\, (\mathcal{D}(Q), \|\cdot\|_{Q,X}))}
\end{aligned}
$$

for all $t \in [0, t_1]$, with

$$
c_1 \quad := \quad \sqrt{\|\hat{P}_1\|_{\mathcal{L}(X, \|\cdot\|_X)}}\,, \tag{6.16}
$$

$$
c_2 \quad := \quad c_1 \sqrt{\|\hat{P}_1^{-1}\|_{\mathcal{L}(X, \|\cdot\|_X)}} \tag{6.17}
$$

and $\|.\|_1$ as in (6.6). So

$$
\begin{aligned}
&\left\| G(\hat{\Pi}+\Pi) - G(\hat{\Pi}) - DG(\hat{\Pi})\Pi \right\|_{C([0,t_1],(X,\|\cdot\|_X))} \\
&\qquad\qquad \leq \quad t_1\, c_2 \, \|\Pi\|_1 \left\| G(\hat{\Pi}+\Pi) - G(\hat{\Pi}) \right\|_{C([0,t_1],(\mathcal{D}(Q), \|\cdot\|_{Q,X}))} .
\end{aligned}
$$

Now we divide by $\|\Pi\|_1$ and take the limit $\Pi \to 0$ which by Proposition 89 exists and is equal to 0.

It remains to prove that $DG(\hat{\Pi})$ is bounded. Therefore we estimate

$$
\begin{aligned}
&\left\| \big[DG(\hat{\Pi})\Pi\big](t) \right\|_X \\
&\quad = \quad \left\| \, S_{\beta(\hat{P})}(t)\, w_0 \; + \int_0^t S_{\beta(\hat{P})}(t-s) \left[f(s) - \beta(P) G(\hat{\Pi})(s) \right] ds \, \right\|_X \\
&\quad \leq \quad c_1 \left\| \, S_{\beta(\hat{P})}(t)\, w_0 \; + \int_0^t S_{\beta(\hat{P})}(t-s) \left[f(s) - \beta(P) G(\hat{\Pi})(s) \right] ds \, \right\|_{E,\hat{P}} \\
&\quad \leq \quad c_1 \Big(\|w_0\|_{E,\hat{P}} \; + \int_0^t \left\| f(s) \right\|_{E,\hat{P}} ds \; + \int_0^t \left\| \beta(P) G(\hat{\Pi})(s) \right\|_{E,\hat{P}} ds \Big) \\
&\quad \leq \quad c_2 \Big(\|w_0\|_X \; + \int_0^t \left\| f(s) \right\|_X ds \; + \int_0^t \left\| \beta(P) G(\hat{\Pi})(s) \right\|_X ds \Big) \\
&\quad \leq \quad c_2 \Big(\|w_0\|_X \; + \; \|f\|_{W^{1,1}((0,t_1),(X,\|\cdot\|_X))} \; + \int_0^{t_1} \left\| G(\hat{\Pi})(s) \right\|_{Q,X} ds \; \|P\|_{\mathcal{T}U,X} \Big)
\end{aligned}
$$

for every $t \in [0, t_1]$ and every element $\Pi = (P, w_0, f)$ of the domain of $DG(\hat{\Pi})$, and where c_1 and c_2 have been defined in (6.16) and (6.17). So

$$\big\|DG(\hat{\Pi})\Pi\big\|_{C([0,t_1],(X,\|.\|_X))} \quad \leq \quad c_2 \max\Big\{1\,,\int_0^{t_1} \big\|G(\hat{\Pi})(s)\big\|_{Q,X}\,ds\Big\}\,\big\|(P, w_0, f)\big\|_1$$

for every element $\Pi = (P, w_0, f)$ of the domain of $DG(\hat{\Pi})$.

Thus $DG(\hat{\Pi})$ actually is the Fréchet-derivative of G at point $\hat{\Pi}$. $\square$

Although for any $\hat{\Pi} = (\hat{P}, \hat{w}_0, \hat{f}) \in U \times \mathcal{D}(Q) \times W^{1,1}\big((0, t_1), X\big)$, the classical solution $G(\hat{\Pi})$ of

$$\begin{aligned} w'(t) \;+\; \beta(\hat{P})\, w(t) \quad &= \quad \hat{f}(t)\,, \qquad\qquad t \in [0, t_1]\,, \\ w(0) \quad &= \quad \hat{w}_0 \end{aligned}$$

is an element of the space $C^1\big([0, t_1], X\big) \cap C\big([0, t_1], \mathcal{D}(Q)\big)$, we cannot expect the function $DG(\hat{\Pi})\Pi$ with $\Pi \in \mathcal{T}U \times \mathcal{D}(Q) \times W^{1,1}\big((0, t_1), X\big)$ to lie in this space, too, as Example 92 will show. That is why the map G has to be interpreted with codomain $C\big([0, t_1], X\big)$ in (6.13).

To formulate this example, we need the following lemma.

Lemma 91. *Let $P \in U$ and $w_0 \in \mathcal{D}(Q)$. Then there is $\delta \in (0, 1)$, such that $\alpha P \in U$ for every $\alpha \in (1 - \delta, 1 + \delta)$, and*

$$G(\alpha P, w_0, 0)(t) \quad = \quad G(P, w_0, 0)(\alpha t)\,, \qquad\qquad t \in \Big[0\,,\, \min\big\{t_1,\, \frac{t_1}{\alpha}\big\}\Big]\,. \tag{6.18}$$

Proof. For $P \in U$ and every $\alpha \in \mathbb{R}$, it is $\alpha P \in \mathcal{T}U$, since $U \subseteq \mathcal{T}U$ and $\mathcal{T}U$ is a vector space. Furthermore, U is open in $\big(\mathcal{T}U, \|.\|_{\mathcal{T}U,X}\big)$, and the mapping $(0, \infty) \to \mathcal{T}U$, $\alpha \mapsto \alpha P$ is continuous. So $\{\alpha \in (0, \infty) : \alpha P \in U\}$ is open and contains 1, which proves the existence of a δ with the properties mentioned above.

To prove the second statement, we fix $w_0 \in \mathcal{D}(Q)$ and $\alpha \in (1 - \delta, 1 + \delta)$ and define $w(t) := G(P, w_0, 0)(\alpha t)$, $t \in [0,\, t_1/\alpha]$. Since $G(P, w_0, 0)$ satisfies

$$G(P, w_0, 0)'(t) \;=\; -\beta(P)\, G(P, w_0, 0)(t)\,, \quad t \in [0, t_1]\,, \qquad G(P, w_0, 0)(0) \;=\; w_0\,,$$

it is

$$\begin{aligned} w'(t) \quad &= \quad \alpha\, G(P, w_0, 0)'(\alpha t) \quad = \quad -\alpha\, \beta(P)\, G(P, w_0, 0)(\alpha t) \\ &= \quad -\alpha\, \beta(P)\, w(t) \quad = \quad -\beta(\alpha P)\, w(t)\,, \end{aligned} \tag{6.19}$$

$t \in [0,\, t_1/\alpha]$, and

$$w(0) \;=\; G(P, w_0, 0)(0) \;=\; w_0\,. \tag{6.20}$$

According to Lemma 5 with t_1 substituted by $\min\{t_1,\, t_1/\alpha\}$, the classical solution of (6.19), (6.20) is unique on $\big[0\,,\, \min\{t_1,\, t_1/\alpha\}\big]$. Thus (6.18) holds true. $\square$

Example 92. Let $P \in U$ and $w_0 \in \mathcal{D}(Q)$. Then for $\lambda > -1$, it holds

$$\begin{aligned}
&\left\| \frac{1}{\lambda}\Big[G\big((1+\lambda)P, w_0, 0\big) - G(P, w_0, 0)\Big] - DG(P, w_0, 0)(P, 0, 0) \right\|_{C([0,t_1],(X,\|.\|_X))} \\
&= \quad \big\|(P,0,0)\big\|_1 \, \frac{1}{\big\|(\lambda P, 0, 0)\big\|_1} \Big\| G(P + \lambda P, w_0, 0) - G(P, w_0, 0) \\
&\qquad\qquad - DG(P, w_0, 0)(\lambda P, 0, 0) \Big\|_{C([0,t_1],(X,\|.\|_X))} \\
&\qquad \longrightarrow 0\,, \qquad \lambda \to 0\,.
\end{aligned}$$

So in particular for any fixed $t \in [0, t_1]$ it is

$$\begin{aligned}
&\lim_{\lambda \to 0} \quad \left\| \frac{1}{\lambda}\Big[G\big((1+\lambda)P, w_0, 0\big)(t) - G(P, w_0, 0)(t)\Big] - \big[DG(P, w_0, 0)(P, 0, 0)\big](t) \right\|_X \\
&= \quad 0\,.
\end{aligned}$$

Hence $g_t(\alpha) \coloneqq G(\alpha P, w_0, 0)(t),\ \alpha \in (1-\delta, 1+\delta)$, where δ is like in Lemma 91, is differentiable in $\alpha = 1$ with the derivative $g_t'(1) = \big[DG(P, w_0, 0)(P, 0, 0)\big](t)$. By (6.18) we have $g_t(\alpha) = G(P, w_0, 0)(\alpha t),\ \alpha \in (1-\delta, 1+\delta) \cap (0,\, t_1/t]$. Therefore we can conclude

$$\begin{aligned}
\big[DG(P, w_0, 0)(P, 0, 0)\big](t) \quad &= \quad g_t'(1) \quad = \quad t\, G(P, w_0, 0)'(\alpha t)\Big|_{\alpha=1} \\
&= \quad t\, G(P, w_0, 0)'(t)\,,
\end{aligned}$$

$t \in [0, t_1]$. Thus if the initial value $w_0 \in \mathcal{D}(Q) \setminus \mathcal{D}\big(\beta(P)^2\big)$, where $\mathcal{D}\big(\beta(P)^2\big) = \{\tilde{w}_0 \in \mathcal{D}(Q) :\ \beta(P)\tilde{w}_0 \in \mathcal{D}(Q)\}$, it is $G(P, w_0, 0) \in C^1\big([0, t_1], X\big) \cap C\big([0, t_1], \mathcal{D}(Q)\big)$. But if we consider the case where $\beta(P)$ even generates a group (instead of merely a semi-group) for example, the function $G(P, w_0, 0)' = -\beta(P)G(P, w_0, 0)$ is not differentiable in any $t_0 \in [0, t_1]$. Otherwise it would also be differentiable in $t = 0$, which in the case $t_0 > 0$ can be seen by time reversal since $\big(S_{\beta(P)}(t)\big)_{t \in \mathbb{R}}$ is assumed to be a group. And this contradicts $\beta(P)w_0 \notin \mathcal{D}(Q)$. It follows that the function $DG(P, w_0, 0)(P, 0, 0)$ is continuous on $[0, t_1]$ but not differentiable in any $t_0 \in (0, t_1]$ either. $\square$

6.1.2 Back Transformation

In section 6.1.1 we proved, that the solution w of the evolution equation (6.5), which we write as

$$\begin{aligned}
w'(t) \quad &= \quad -\beta(P)\, w(t) \ + \ g(t)\,, \qquad\qquad t \in [0, t_1]\,, \\
w(0) \quad &= \quad w_0\,,
\end{aligned} \tag{6.21}$$

where we denote the inhomogeneity by g now, can be differentiated for the material parameters P, the initial value w_0 and the right-hand side g. Motivated by our application we assume now, that (6.21) is the result of an abstract variable transformation $T \in \mathcal{L}(X)$ as studied in section 4.1. More precisely, there is a second evolution equation

$$\begin{aligned} u'(t) &= -A\,u(t) + f(t)\,, & t \in [0, t_1]\,, \\ u(0) &= u_0\,, \end{aligned} \tag{6.22}$$

such that $\beta(P) = TAT^{-1}$, $\mathcal{D}(Q) = T\mathcal{D}(A)$, $g(t) = Tf(t)$, $t \in [0, t_1]$, and $w_0 = Tu_0$. By Theorem 27, a function $u \in C^1\big([0,t_1], X\big) \cap C\big([0,t_1], \mathcal{D}(A)\big)$ solves (6.22), if and only if Tu solves (6.21).

In our application, the variable transformation T depends on the material parameters $P \in U$. So we are going to denote it by $T(P)$. Furthermore, in our application, the map $P \mapsto T(P)$ is differentiable. Now the question arises, how the solution u of (6.22) can be differentiated for the triple (P, u_0, f). The answer to this question is given in this section.

Assumption 93. In addition to Assumption 82 we assume the following in this section.

Let $T : \big(U, \|.\|_{\mathcal{T}U}\big) \to \mathcal{L}\big(X, \|.\|_X\big)$ be Fréchet-differentiable and such that for every $P \in U$ the bounded linear operator $T(P)$ is invertible with inverse $T(P)^{-1} \in \mathcal{L}\big(X, \|.\|_X\big)$. Then the map $U \to \mathcal{L}\big(X, \|.\|_X\big)$, $P \mapsto T(P)^{-1}$ is Fréchet-differentiable, too.

Moreover, we assume that the set-valued map $U \to 2^X$, $P \mapsto T(P)^{-1}\mathcal{D}(Q)$ is constant, and that there exists a linear operator $\tilde{Q} : X \supseteq \mathcal{D}(\tilde{Q}) \to X$ with domain of definition $\mathcal{D}(\tilde{Q})$, such that $\mathcal{D}(\tilde{Q}) = T(P)^{-1}\mathcal{D}(Q)$ for all $P \in U$, furthermore $\big[DT(\hat{P})^{-1}P\big]\mathcal{D}(Q) \subseteq \mathcal{D}(\tilde{Q})$, $\hat{P} \in U$, $P \in \mathcal{T}U$, and that for every $\hat{P} \in U$ there is a neighborhood $W \subseteq U$ and constants $c, C > 0$ such that for the graph norms $\|.\|_{\tilde{Q},X} := \|\tilde{Q}\,.\,\|_X + \|.\|_X$ and $\|.\|_{T(P)^{-1}\beta(P)T(P),\,X} := \|T(P)^{-1}\beta(P)T(P)\,.\,\|_X + \|.\|_X$ on $\mathcal{D}(\tilde{Q})$ it holds

$$c\,\|u\|_{\tilde{Q},X} \quad \le \quad \|u\|_{T(P)^{-1}\beta(P)T(P),\,X} \quad \le \quad C\,\|u\|_{\tilde{Q},X}\,, \qquad u \in \mathcal{D}(\tilde{Q})\,, \quad P \in W\,.$$

We assume $\mathcal{D}(\tilde{Q})$ to be endowed with the graph norm $\|.\|_{\tilde{Q},X}$, analogously to $\mathcal{D}(Q)$ being equipped with $\|.\|_{Q,X} = \|Q\,.\,\|_X + \|.\|_X$. □

Notation 94. To simplify our notation, we do not distinguish between the two maps $T(P) \in \mathcal{L}(X)$ and $\tilde{T}(P) \in \mathcal{L}\big(C([0,t_1], X)\big)$ with $\big(\tilde{T}(P)u\big)(t) := T(P)\big(u(t)\big)$, where $P \in U$, any more, in contrast to section 4.1. In this section, both shall be denoted by $T(P)$. □

From Lemma 26 together with Lemma 24, Lemma 83, and the statement on norm equivalences in Assumption 93, it follows that

$$T(P)^{-1}\, w \quad \in \quad C^1\big([0,t_1],X\big)\cap C\big([0,t_1],\mathcal{D}(\tilde{Q})\big)$$

for every $w\in C^1\big([0,t_1],X\big)\cap C\big([0,t_1],\mathcal{D}(Q)\big)$.

Definition 95. *As the* ***back-transformed abstract parameter-to-solution-map*** *(with respect to T), we denote the map*

$$\begin{aligned} H:\ U\times\mathcal{D}(\tilde{Q})\times W^{1,1}\big((0,t_1),X\big) \quad &\to \quad C^1\big([0,t_1],X\big)\cap C\big([0,t_1],\mathcal{D}(\tilde{Q})\big)\,,\\ H(P,u_0,f)(t) \quad &= \quad T(P)^{-1}\, G\Big(P,\, T(P)u_0,\, T(P)f\Big)(t)\,, \end{aligned}$$

where G is the abstract parameter-to-solution-map defined in (6.4). □

We recall that for any $\hat{\Pi}:=(\hat{P},\hat{w}_0,\hat{g})\in U\times\mathcal{D}(Q)\times W^{1,1}\big((0,t_1),X\big)$, it holds $G(\hat{\Pi})\in C^1\big([0,t_1],X\big)\cap C\big([0,t_1],\mathcal{D}(Q)\big)$ and

$$\begin{aligned} G(\hat{\Pi})'(t) \quad &= \quad -\beta(\hat{P})\, G(\hat{\Pi})(t)\;+\;\hat{g}(t)\,, \qquad t\in[0,t_1]\,,\\ G(\hat{\Pi})(0) \quad &= \quad \hat{w}_0\,. \end{aligned}$$

So according to Theorem 27, for any $\hat{\Pi}:=(\hat{P},\hat{u}_0,\hat{f})\in U\times\mathcal{D}(\tilde{Q})\times W^{1,1}\big((0,t_1),X\big)$, the function $H(\hat{\Pi})$ satisfies

$$\begin{aligned} H(\hat{\Pi})'(t) \quad &= \quad -T(\hat{P})^{-1}\beta(\hat{P})T(\hat{P})\, H(\hat{\Pi})(t)\;+\;\hat{f}(t)\,, \qquad t\in[0,t_1]\,,\\ H(\hat{\Pi})(0) \quad &= \quad \hat{u}_0 \end{aligned}$$

in the classical sense.

Furthermore, $G:\big(U,\|.\|_{\mathcal{T}U}\big)\times\big(\mathcal{D}(Q),\|.\|_{Q,X}\big)\times W^{1,1}\big((0,t_1),X\big)\to C\big([0,t_1],X\big)$ is Fréchet-differentiable, and for $\hat{\Pi}:=(\hat{P},\hat{w}_0,\hat{g})\in U\times\mathcal{D}(Q)\times W^{1,1}\big((0,t_1),X\big)$ and $\Pi:=(P,w_0,g)\in\mathcal{T}U\times\mathcal{D}(Q)\times W^{1,1}\big((0,t_1),X\big)$, it is

$$\big[DG(\hat{\Pi})\Pi\big](t)\;=\;S_{\beta(\hat{P})}(t)\,w_0\;+\;\int_0^t S_{\beta(\hat{P})}(t-s)\,\big[g(s)-\beta(P)G(\hat{\Pi})(s)\big]\,ds\,,$$

$t\in[0,t_1]$, which is the mild solution of

$$\begin{aligned} w'(t)\;+\;\beta(\hat{P})\,w(t) \quad &= \quad g(t)\;-\;\beta(P)\,G(\hat{\Pi})(t)\,, \qquad t\in[0,t_1]\,,\\ w(0) \quad &= \quad w_0\,. \end{aligned}$$

The following two lemmas are useful auxiliaries for differentiating functions, which are compositions of several other functions.

Lemma 96. *(Product Rule) Let $\big(Y, \|.\|_Y\big)$, $\big(Z, \|.\|_Z\big)$ be normed spaces, $V \subseteq Y$ open and $\tau : V \to \mathcal{L}(Z)$, $\nu : V \to Z$ Fréchet-differentiable. Then also the map $V \to Z$, $y \mapsto \tau(y)\nu(y)$ is Fréchet-differentiable, and its derivative at a point $\hat{y} \in V$ evaluated in a direction $y \in Y$ is of the form $\big[D\tau(\hat{y})y\big]\nu(\hat{y}) + \tau(\hat{y})\big[D\nu(\hat{y})y\big]$.*

Proof. For fixed $\hat{y} \in V$ and every $y \in Y$ with $\hat{y} + y \in V$, it is

$$\begin{aligned}
&\frac{\Big\|\tau(\hat{y}+y)\nu(\hat{y}+y) - \tau(\hat{y})\nu(\hat{y}) - \big[D\tau(\hat{y})y\big]\nu(\hat{y}) - \tau(\hat{y})\big[D\nu(\hat{y})y\big]\Big\|_Z}{\|y\|_Y} \\
\leq \quad & \frac{\big\|\tau(\hat{y}+y) - \tau(\hat{y}) - D\tau(\hat{y})y\big\|_{\mathcal{L}(Z)}}{\|y\|_Y} \big\|\nu(\hat{y}+y)\big\|_Z \\
+ \quad & \big\|\tau(\hat{y})\big\|_{\mathcal{L}(Z)} \frac{\big\|\nu(\hat{y}+y) - \nu(\hat{y}) - D\nu(\hat{y})y\big\|_Z}{\|y\|_Y} \\
+ \quad & \frac{\big\|D\tau(\hat{y})\big\|_{\mathcal{L}(Y,Z)}\|y\|_Y\big\|\nu(\hat{y}+y) - \nu(\hat{y})\big\|_Z}{\|y\|_Y},
\end{aligned}$$

which tends to 0 for $y \to 0$, since τ and ν are differentiable and therefore also continuous. □

Lemma 97. *Let J, K be finite sets, $\big(Y_j, \|.\|_{Y_j}\big)$, $\big(Z_k, \|.\|_{Z_k}\big)$ denote normed spaces and $V_j \subseteq Y_j$ open sets for $j \in J$ and $k \in K$, and let $\iota : K \to 2^J$ be a (set valued) map. Let $\gamma_k : \bigtimes_{j\in\iota(k)} V_j \to Z_k$ for $k \in K$ be Fréchet-differentiable maps. Then also the map*

$$\gamma : \bigtimes_{j\in J} \big(V_j, \|.\|_{Y_j}\big) \to \bigtimes_{k\in K} \big(Z_k, \|.\|_{Z_k}\big), \qquad (y_j)_{j\in J} \mapsto \Big(\gamma_k\big((y_j)_{j\in\iota(k)}\big)\Big)_{k\in K}$$

is Fréchet-differentiable, and its derivative at a point $(\hat{y}_j)_{j\in J} \in \bigtimes_{j\in J} V_j$ in a direction $(\hat{y}_j)_{j\in J} \in \bigtimes_{j\in J} Y_j$ is given by $\Big(D\gamma_k\big((\hat{y}_j)_{j\in\iota(k)}\big)(y_j)_{j\in\iota(k)}\Big)_{k\in K}$.

Proof. Let $(\hat{y}_j)_{j\in J} \in \bigtimes_{j\in J} V_j$ and $(y_j)_{j\in J} \in \bigtimes_{j\in J} Y_j$, such that $(\hat{y}_j)_{j\in J} + (y_j)_{j\in J} \in \bigtimes_{j\in J} V_j$. Then

$$\begin{aligned}
&\frac{1}{\big\|(y_j)_{j\in J}\big\|_1} \bigg\| \Big(\gamma_k\big((\hat{y}_j)_{j\in\iota(k)} + (y_j)_{j\in\iota(k)}\big)\Big)_{k\in K} - \Big(\gamma_k\big((\hat{y}_j)_{j\in\iota(k)}\big)\Big)_{k\in K} \\
&\qquad\qquad - \Big(D\gamma_k\big((\hat{y}_j)_{j\in\iota(k)}\big)(y_j)_{j\in\iota(k)}\Big)_{k\in K}\bigg\|_1 \\
= \quad & \frac{1}{\sum_{j\in J}\|y_j\|_{Y_j}} \sum_{k\in K} \Big\|\gamma_k\big((\hat{y}_j)_{j\in\iota(k)} + (y_j)_{j\in\iota(k)}\big) - \gamma_k\big((\hat{y}_j)_{j\in\iota(k)}\big)
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad - D\gamma_k\big((\hat{y}_j)_{j\in\iota(k)}\big)(y_j)_{j\in\iota(k)}\Big\|_{Z_k} \\
\leq\ & \sum_{k\in K} \frac{1}{\sum_{j\in\iota(k)}\|y_j\|_{Y_j}} \Big\|\gamma_k\big((\hat{y}_j)_{j\in\iota(k)} + (y_j)_{j\in\iota(k)}\big) - \gamma_k\big((\hat{y}_j)_{j\in\iota(k)}\big) \\
&\qquad\qquad\qquad\qquad - D\gamma_k\big((\hat{y}_j)_{j\in\iota(k)}\big)(y_j)_{j\in\iota(k)}\Big\|_{Z_k} \\
=\ & \sum_{k\in K} \frac{1}{\big\|(y_j)_{j\in\iota(k)}\big\|_1} \Big\|\gamma_k\big((\hat{y}_j)_{j\in\iota(k)} + (y_j)_{j\in\iota(k)}\big) - \gamma_k\big((\hat{y}_j)_{j\in\iota(k)}\big) \\
&\qquad\qquad\qquad\qquad - D\gamma_k\big((\hat{y}_j)_{j\in\iota(k)}\big)(y_j)_{j\in\iota(k)}\Big\|_{Z_k} \\
&\to 0\,, \qquad (y_j)_{j\in J} \to 0\,,
\end{aligned}
$$

since γ_k is differentiable, $k \in K$. □

Lemma 98. *The map*

$$
\begin{aligned}
\tilde{G} : \big(U, \|.\|_{\mathcal{T}U,X}\big) \times \big(\mathcal{D}(\tilde{Q}), \|.\|_{\tilde{Q},X}\big) \times W^{1,1}\big((0,t_1), X\big) &\quad\to\quad C\big([0,t_1], X\big) \\
(P, u_0, f) &\quad\mapsto\quad G\big(P, T(P)u_0, T(P)f\big)
\end{aligned}
$$

is Fréchet-differentiable with derivative

$$
\begin{aligned}
&\big[D\tilde{G}(\hat{P}, \hat{u}_0, \hat{f})(P, u_0, f)\big](t) \\
&\quad = \quad S_{\beta(\hat{P})}(t)\,\Big(\big[DT(\hat{P})P\big]\hat{u}_0 + T(\hat{P})u_0\Big) \\
&\qquad\qquad + \int_0^t S_{\beta(\hat{P})}(t-s)\,\Big(\big[DT(\hat{P})P\big]\hat{f}(s) + T(\hat{P})f(s) \\
&\qquad\qquad\qquad\qquad\qquad\qquad - \beta(P)\tilde{G}(\hat{P}, \hat{u}_0, \hat{f})(s)\Big)\,ds\,,
\end{aligned}
$$

$t \in [0, t_1]$, *which is the mild solution of*

$$
\begin{aligned}
w'(t) &= -\beta(\hat{P})\,w(t) - \beta(P)\hat{w}(t) + \big[DT(\hat{P})P\big]\hat{f}(t) + T(\hat{P})f(t)\,, \quad t \in [0, t_1]\,, \\
w(0) &= \big[DT(\hat{P})P\big]\hat{u}_0 + T(\hat{P})u_0\,,
\end{aligned}
$$

where $\hat{w}$ *satisfies*

$$
\begin{aligned}
\hat{w}'(t) &= -\beta(\hat{P})\,\hat{w}(t) \ +\ T(\hat{P})\hat{f}(t)\,, \qquad t \in [0, t_1]\,, \\
\hat{w}(0) &= T(\hat{P})\hat{u}_0
\end{aligned}
$$

in the classical sense.

Proof. This is proven by applying Lemma 96 to the arguments of G and using Lemma 97, the differentiability of G and the chain rule. □

Theorem 99. *Interpreted with the bigger codomain* $C\big([0,t_1]),X\big)$, *the map* H *is Fréchet-differentiable. For fixed* $\hat\Pi := (\hat P, \hat u_0, \hat f) \in U \times \mathcal{D}(\tilde Q) \times W^{1,1}\big((0,t_1),X\big)$ *and* $\Pi := (P, u_0, f) \in \mathcal{T}U \times \mathcal{D}(\tilde Q) \times W^{1,1}\big((0,t_1),X\big)$, *we abbreviate* $T := T(\hat P)$, $T^{-1} := T(\hat P)^{-1}$, $T_1 := DT(\hat P)^{-1}P$,

$$\hat A \quad := \quad T^{-1}\beta(\hat P)T$$

and

$$A \quad := \quad T_1\beta(\hat P)T \;-\; \hat A T_1 T \;+\; T^{-1}\beta(P)T\,.$$

Then

$$\big[DH(\hat\Pi)\Pi\big](t) \quad = \quad S_{\hat A}(t)u_0 \;+\; \int_0^t S_{\hat A}(t-s)\big(-A\hat u(s) + f(s)\big)\,ds\,,$$

$t \in [0,t_1]$, *where* $\hat u$ *is the classical solution of*

$$\begin{aligned} \hat u'(t) \quad &= \quad -\hat A \hat u(t) \;+\; \hat f(t)\,, \qquad\qquad t \in [0,t_1]\,, \\ \hat u(0) \quad &= \quad \hat u_0\,. \end{aligned}$$

In other words, $DH(\hat\Pi)\Pi$ *is the mild solution of*

$$\begin{aligned} u'(t) \quad &= \quad -\hat A u(t) \;-\; A\hat u(t) \;+\; f(t)\,, \qquad\qquad t \in [0,t_1]\,, \\ u(0) \quad &= \quad u_0\,. \end{aligned}$$

Proof. That H is differentiable, can be shown by applying Lemma 96 together with Lemma 97 and Lemma 98. Let $\hat\Pi := (\hat P, \hat u_0, \hat f) \in U \times \mathcal{D}(\tilde Q) \times W^{1,1}\big((0,t_1),X\big)$ and $\Pi := (P, u_0, f) \in \mathcal{T}U \times \mathcal{D}(\tilde Q) \times W^{1,1}\big((0,t_1),X\big)$. According to Lemma 96, it is

$$\big[DH(\hat\Pi)\Pi\big](t) \quad = \quad \big[DT(\hat P)^{-1}P\big]\tilde G(\hat\Pi)(t) \;+\; T(\hat P)^{-1}\big[D\tilde G(\hat\Pi)\Pi\big](t)\,, \tag{6.23}$$

$t \in [0,t_1]$, with $\tilde G$ from Lemma 98.

For better readability we use the abbreviations

$$\begin{aligned} &T \;=\; T(\hat P)\,, \qquad T^{-1} \;=\; T(\hat P)^{-1}\,, \qquad T_1 \;=\; DT(\hat P)^{-1}P\,, \\ &\hat A \;=\; T^{-1}\beta(\hat P)T\,, \\ &\hat w \;:=\; \tilde G(\hat\Pi)\,, \qquad w := D\tilde G(\hat\Pi)\Pi\,, \\ &\hat u \;:=\; H(\hat\Pi) \;=\; T^{-1}\hat w\,, \qquad\qquad u \;:=\; DH(\hat\Pi)\Pi \;=\; T_1\hat w + T^{-1}w\,, \\ &\hat u_0 := H(\hat\Pi)(0)\,, \qquad u_0 \;:=\; u(0) \;=\; T_1\hat w(0) + T^{-1}w(0)\,, \\ &g \;:=\; \big[DT(\hat P)P\big]\hat f + Tf\,. \end{aligned}$$

In the sequel, we use (6.23), Lemma 98 and Corollary 30, which verifies the formula $T^{-1}S_{\beta(\hat{P})}(\cdot)T = S_{\hat{A}}(\cdot)$, to find

$$\begin{aligned}
\big[DH(\hat{\Pi})\Pi\big](t) \;&=\; T_1\hat{w}(t) \;+\; T^{-1}w(t)\\
&=\; T_1\hat{w}(t) \;+\; T^{-1}\Big(S_{\beta(\hat{P})}(t)\,w(0)\\
&\qquad\qquad + \int_0^t S_{\beta(\hat{P})}(t-s)\,\big[g(s)-\beta(P)\hat{w}(s)\big]\,ds\Big)\\
&=\; T_1\hat{w}(t) \;+\; T^{-1}S_{\beta(\hat{P})}(t)T\,T^{-1}w(0)\\
&\qquad\qquad + \int_0^t T^{-1}S_{\beta(\hat{P})}(t-s)T\,T^{-1}\big[g(s)-\beta(P)\hat{w}(s)\big]\,ds\\
&=\; T_1\hat{w}(t) \;+\; S_{\hat{A}}(t)\,T^{-1}w(0)\\
&\qquad\qquad + \int_0^t S_{\hat{A}}(t-s)\,T^{-1}\big[g(s)-\beta(P)\hat{w}(s)\big]\,ds\\
&=\; T_1\hat{w}(t) \;-\; S_{\hat{A}}(t)\,T_1\hat{w}(0) \;+\; S_{\hat{A}}(t)\,u_0\\
&\qquad\qquad + \int_0^t S_{\hat{A}}(t-s)\,\big[T^{-1}g(s)-T^{-1}\beta(P)\hat{w}(s)\big]\,ds\,,
\end{aligned}$$

$t\in[0,t_1]$. In the following, we focus on the first two summands of this result. With $S_{\hat{A}}(0)=\mathrm{Id}$, $\frac{d}{dt}S_{\hat{A}}(t)v = -S_{\hat{A}}(t)\hat{A}v$ for $t\in[0,\infty)$ and $v\in\mathcal{D}(\tilde{Q})$, $T_1\mathcal{D}(Q)\subseteq\mathcal{D}(\tilde{Q})$, and the fundamental theorem of calculus, it is

$$\begin{aligned}
&T_1\hat{w}(t) \;-\; S_{\hat{A}}(t)\,T_1\hat{w}(0)\\
&\quad=\; S_{\hat{A}}(t-t)T_1\hat{w}(t) \;-\; S_{\hat{A}}(t-0)\,T_1\hat{w}(0)\\
&\quad=\; \int_0^t \frac{d}{ds}S_{\hat{A}}(t-s)T_1\hat{w}(s)\,ds\\
&\quad=\; \int_0^t S_{\hat{A}}(t-s)\hat{A}T_1\hat{w}(s)+S_{\hat{A}}(t-s)T_1\hat{w}'(s)\,ds\\
&\quad=\; \int_0^t S_{\hat{A}}(t-s)\Big(\hat{A}T_1\hat{w}(s)-T_1\beta(\hat{P})\hat{w}(s)+T_1T\hat{f}(s)\Big)\,ds\,,
\end{aligned}$$

$t\in[0,t_1]$. Altogether, we get

$$\begin{aligned}
&\big[DH(\hat{\Pi})\Pi\big](t)\\
&\quad=\; S_{\hat{A}}(t)u_0\\
&\qquad + \int_0^t S_{\hat{A}}(t-s)\Big(-T_1\beta(\hat{P})\hat{w}(s)+\hat{A}T_1\hat{w}(s)-T^{-1}\beta(P)\hat{w}(s)\\
&\qquad\qquad\qquad + T_1T\hat{f}(s)+T^{-1}g(s)\Big)\,ds
\end{aligned}$$

$$
\begin{aligned}
&= S_{\hat{A}}(t)u_0 \\
&\quad + \int_0^t S_{\hat{A}}(t-s)\Big(- T_1\beta(\hat{P})T\hat{u}(s) + \hat{A}T_1T\hat{u}(s) - T^{-1}\beta(P)T\hat{u}(s) \\
&\qquad\qquad + f(s)\Big)\, ds\,,
\end{aligned}
$$

$t \in [0, t_1]$. In the last step, we applied Lemma 96 again by calculating

$$
\begin{aligned}
T_1T\hat{f} \,+\, T^{-1}g &= \big[DT(\hat{P})^{-1}P\big]T(\hat{P})\hat{f} \,+\, T(\hat{P})^{-1}\Big(\big[DT(\hat{P})P\big]\hat{f} + T(\hat{P})f\Big) \\
&= \Big(\big[DT(\hat{P})^{-1}P\big]T(\hat{P}) + T(\hat{P})^{-1}\big[DT(\hat{P})P\big]\Big)\hat{f} \,+\, f \\
&= \big[D\big(T(\hat{P})^{-1}T(\hat{P})\big)P\big]\hat{f} \,+\, f \\
&= \big[D\mathrm{Id}_{\mathcal{L}(X)}P\big]\hat{f} \,+\, f \\
&= 0 \,+\, f \\
&= f\,.
\end{aligned}
$$

□

6.2 Viscoelasticity

In this section we prove the differentiability of the solution of (5.23), that is

$$
\begin{aligned}
\mathbf{v}'(t) &= \vartheta \operatorname{div} \boldsymbol{\sigma}(t) \,+\, \vartheta\, \mathbf{f}(t)\,, \\
\boldsymbol{\sigma}'(t) &= C\Big(\mu_H + \sum_{l=1}^{L} \mu_{M,l}\,,\; \kappa_H + \sum_{l=1}^{L} \kappa_{M,l}\Big)\, \varepsilon\big(\mathbf{v}(t)\big) \,+\, \sum_{l=1}^{L} \boldsymbol{\eta}_l(t) \,+\, \mathbf{g}(t)\,, \\
\boldsymbol{\eta}_l'(t) &= -\omega_{\boldsymbol{\sigma},l} C\big(\mu_{M,l}\,,\, \kappa_{M,l}\big)\, \varepsilon\big(\mathbf{v}(t)\big) \,-\, \omega_{\boldsymbol{\sigma},l}\boldsymbol{\eta}_l(t)\,, \qquad l = 1, \ldots, L\,,
\end{aligned}
\tag{6.24}
$$

$$
\mathbf{v}(0) \,=\, \mathbf{v}^{(0)}\,, \qquad \boldsymbol{\sigma}(0) \,=\, \boldsymbol{\sigma}^{(0)}\,, \qquad \boldsymbol{\eta}(0) \,=\, \boldsymbol{\eta}^{(0)}\,,
$$
$$
\mathbf{v}(t)\big|_{\partial D_D} \,=\, \mathbf{0}\,, \qquad \mathbf{n}^\top \boldsymbol{\sigma}(t)\big|_{\partial D_N} \,=\, \mathbf{0}\,,
$$

$t \in [0, t_1]$, with respect to the material parameters $\vartheta, \ldots, \omega_{\boldsymbol{\sigma},L}$ as well as the inhomogeneity $\mathbf{f}$, $\mathbf{g}$ and the initial values $\mathbf{v}^{(0)}, \boldsymbol{\sigma}^{(0)}, \boldsymbol{\eta}^{(0)}$. As we have seen before, we can write this initial-boundary value problem as the evolution equation

$$
\begin{aligned}
u'(t) &= -Au(t) \,+\, f(t)\,, \qquad\qquad t \in [0, t_1]\,, \\
u(0) &= u_0
\end{aligned}
$$

with right-hand side $f = (\vartheta \mathbf{f}, \mathbf{g}, \mathbf{0})^\top$, initial value $u_0 = (\mathbf{v}^{(0)}, \boldsymbol{\sigma}^{(0)}, \boldsymbol{\eta}^{(0)})^\top$ and generator

$$A \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma} \\ \boldsymbol{\eta} \end{pmatrix} = - \begin{pmatrix} \vartheta \operatorname{div} \boldsymbol{\sigma} \\ C\Big(\mu_H + \sum_{l=1}^{L} \mu_{M,l}\,,\ \kappa_H + \sum_{l=1}^{L} \kappa_{M,l}\Big)\, \varepsilon(\mathbf{v}) \;+\; \sum_{l=1}^{L} \boldsymbol{\eta}_l \\ -\omega_{\boldsymbol{\sigma},1} C\big(\mu_{M,1}\,,\ \kappa_{M,1}\big)\, \varepsilon(\mathbf{v}) \;-\; \omega_{\boldsymbol{\sigma},1} \boldsymbol{\eta}_1 \\ \vdots \\ -\omega_{\boldsymbol{\sigma},L} C\big(\mu_{M,L}\,,\ \kappa_{M,L}\big)\, \varepsilon(\mathbf{v}) \;-\; \omega_{\boldsymbol{\sigma},L} \boldsymbol{\eta}_L \end{pmatrix}, \quad (6.25)$$

$(\mathbf{v}, \boldsymbol{\sigma}, \boldsymbol{\eta})^\top \in \mathcal{D}(A)$, on the domain of definition

$$\mathcal{D}(A) \quad = \quad V \times S \times L^2(D, \mathbb{R}^{3\times 3}_{\mathrm{sym}})^L \quad (6.26)$$

from (5.11) with

$$V \quad = \quad \overline{\Big\{\, \boldsymbol{\varphi} \in C^\infty(D, \mathbb{R}^3) \cap H(\varepsilon, D, \mathbb{R}^3) \;:\quad \partial D_D \subseteq \mathbb{R}^3 \setminus \operatorname{supp}(\boldsymbol{\varphi}) \,\Big\}}^{\|\cdot\|_V}$$

and

$$S \;=\; \Big\{\, \boldsymbol{\sigma} \in H(\operatorname{div}, D, \mathbb{R}^{3\times 3}_{\mathrm{sym}}) :\ \forall \boldsymbol{\varphi} \in V :\ \int_D \varepsilon(\boldsymbol{\varphi}) : \boldsymbol{\sigma} \;+\; \boldsymbol{\varphi} \cdot \operatorname{div} \boldsymbol{\sigma}\, dx = 0 \,\Big\}.$$

We are going to derive the differentiability of the solution of (6.24) indirectly by transforming (6.24) to

$$\begin{aligned} \mathbf{v}'(t) \quad &= \quad \vartheta \operatorname{div} \Big(\boldsymbol{\sigma}_H + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l}\Big)(t) \;+\; \vartheta\, \mathbf{f}(t)\,, \\ \boldsymbol{\sigma}'_H(t) \quad &= \quad C(\mu_H\,,\, \kappa_H)\, \varepsilon\big(\mathbf{v}(t)\big) \;+\; \mathbf{g}(t)\,, \\ \boldsymbol{\sigma}'_{M,l}(t) \quad &= \quad C(\mu_{M,l}\,,\, \kappa_{M,l})\, \varepsilon\big(\mathbf{v}(t)\big) \;-\; \omega_{\boldsymbol{\sigma},l} \boldsymbol{\sigma}_{M,l}(t)\,, \qquad l = 1, \ldots, L\,, \end{aligned} \quad (6.27)$$

$$\mathbf{v}(0) \;=\; \mathbf{v}^{(0)}\,, \qquad \boldsymbol{\sigma}_H(0) \;=\; \boldsymbol{\sigma}^{(0)} + \sum_{l=1}^{L} \frac{1}{\omega_{\boldsymbol{\sigma},l}} \boldsymbol{\eta}_l^{(0)}\,,$$

$$\boldsymbol{\sigma}_{M,l}(0) \;=\; -\frac{1}{\omega_{\boldsymbol{\sigma},l}} \boldsymbol{\eta}_l^{(0)}\,, \qquad l = 1, \ldots, L\,,$$

$$\mathbf{v}(t)\big|_{\partial D_D} \;=\; \mathbf{0}\,, \qquad \mathbf{n}^\top \Big(\boldsymbol{\sigma}_H + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l}\Big)(t)\Big|_{\partial D_N} \;=\; \mathbf{0}\,,$$

$t \in [0, t_1]$ via Theorem 65 and studying the latter initial-boundary value problem. As in the proof of Theorem 65, problem (6.27) can be written as the evolution equation

$$w'(t) \;=\; -B\, w(t) \,+\, f(t)\,, \qquad t \in [0, t_1]\,, \qquad\qquad w(0) \;=\; w_0\,,$$

where

$$B \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma}_H \\ \boldsymbol{\sigma}_{M,1} \\ \vdots \\ \boldsymbol{\sigma}_{M,L} \end{pmatrix} \;=\; - \begin{pmatrix} \vartheta\, \mathrm{div}\left(\boldsymbol{\sigma}_H + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l}\right) \\ C\big(\mu_H, \kappa_H\big)\, \varepsilon\big(\mathbf{v}\big) \\ C\big(\mu_{M,1}, \kappa_{M,1}\big)\, \varepsilon(\mathbf{v}) - \omega_{\boldsymbol{\sigma},1} \boldsymbol{\sigma}_{M,1} \\ \vdots \\ C\big(\mu_{M,L}, \kappa_{M,L}\big)\, \varepsilon(\mathbf{v}) - \omega_{\boldsymbol{\sigma},L} \boldsymbol{\sigma}_{M,L} \end{pmatrix},$$

$(\mathbf{v}, \boldsymbol{\sigma}_H, \boldsymbol{\sigma}_M)^\top \in \mathcal{D}(B)$, as in (5.29),

$$\mathcal{D}(B) \;=\; \Big\{ (\mathbf{v}, \boldsymbol{\sigma}_H, \boldsymbol{\sigma}_M)^\top \in\; V \times L^2(D, \mathbb{R}^{3\times 3}_{\mathrm{sym}}) \times L^2(D, \mathbb{R}^{3\times 3}_{\mathrm{sym}})^L \;: \\ \boldsymbol{\sigma}_H + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l} \in S \Big\}$$

as in (5.28), $f = (\vartheta \mathbf{f}, \mathbf{g}, \mathbf{0})^\top$ and $w_0 = T u_0$ with $u_0 = (\mathbf{v}^{(0)}, \boldsymbol{\sigma}^{(0)}, \boldsymbol{\eta}^{(0)})^\top$ and T defined as in (5.24). That is,

$$T \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma} \\ \boldsymbol{\eta}_1 \\ \vdots \\ \boldsymbol{\eta}_L \end{pmatrix} \;=\; \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma} + \sum_{l=1}^{L} \frac{1}{\omega_{\boldsymbol{\sigma},l}} \boldsymbol{\eta}_l \\ -\frac{1}{\omega_{\boldsymbol{\sigma},1}} \boldsymbol{\eta}_1 \\ \vdots \\ -\frac{1}{\omega_{\boldsymbol{\sigma},L}} \boldsymbol{\eta}_L \end{pmatrix},$$

$(\mathbf{v}, \boldsymbol{\sigma}, \boldsymbol{\eta})^\top \in X$, and

$$T^{-1} \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma}_H \\ \boldsymbol{\sigma}_{M,1} \\ \vdots \\ \boldsymbol{\sigma}_{M,L} \end{pmatrix} \;=\; \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma}_H + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l} \\ -\omega_{\boldsymbol{\sigma},1} \boldsymbol{\sigma}_{M,1} \\ \vdots \\ -\omega_{\boldsymbol{\sigma},L} \boldsymbol{\sigma}_{M,L} \end{pmatrix}, \tag{6.28}$$

$(\mathbf{v}, \boldsymbol{\sigma}_H, \boldsymbol{\sigma}_M)^\top \in X$, as stated in (5.25).

We further recall that by Lemma 59, the operator B can be decomposed as $B = -P_1 Q + P_2$ with $P_1, P_2 : X \to X$,

$$P_1 \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma}_H \\ \boldsymbol{\sigma}_M \end{pmatrix} = \begin{pmatrix} \vartheta\, \mathbf{v} \\ C(\mu_H, \kappa_H)\, \boldsymbol{\sigma}_H \\ C(\mu_{M,1}, \kappa_{M,1})\, \boldsymbol{\sigma}_{M,1} \\ \vdots \\ C(\mu_{M,L}, \kappa_{M,L})\, \boldsymbol{\sigma}_{M,L} \end{pmatrix}, \qquad P_2 \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma}_H \\ \boldsymbol{\sigma}_M \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \omega_{\boldsymbol{\sigma},1} \boldsymbol{\sigma}_{M,1} \\ \vdots \\ \omega_{\boldsymbol{\sigma},L} \boldsymbol{\sigma}_{M,L} \end{pmatrix} \tag{6.29}$$

and $Q : X \supseteq \mathcal{D}(B) \to X$,

$$Q \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma}_H \\ \boldsymbol{\sigma}_M \end{pmatrix} = \begin{pmatrix} \operatorname{div} \Big(\boldsymbol{\sigma}_H + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l} \Big) \\ \varepsilon(\mathbf{v}) \\ \varepsilon(\mathbf{v}) \\ \vdots \\ \varepsilon(\mathbf{v}) \end{pmatrix}.$$

To apply Theorem 90, which guarantees the differentiability of the abstract parameter-to-solution-map from Definition 85, we need to assure the preliminaries listed in Assumption 82. This is done in the next paragraph on notation and the following lemmas up to Corollary 103.

Notation 100. The operator B will take the role from $\beta(P)$ in Assumption 82, and we will also denote it this way. Its domain of definition is independent of P_1 and P_2, so we can define $\mathcal{D}(Q) := \mathcal{D}\big(\beta(P)\big)$.

Next we need concrete instances of U and $\mathcal{T}U$ from Assumption 82. To this end, we introduce the **material parameter set**

$$\mathcal{P} \quad := \quad L^\infty_+(D)^{3+3L} \quad \subseteq \quad L^\infty(D, \mathbb{R}^{3+3L}),$$

where

$$L^\infty_+(D) \;=\; \Big\{ \alpha \in L^\infty(D) \;:\; \alpha > 0,\; \frac{1}{\alpha} \in L^\infty(D) \Big\}$$

has been defined in (5.13). The space $L^\infty(D, \mathbb{R}^{3+3L})$ is assumed to have the norm

$$\big\| (\vartheta, \ldots, \omega_{\boldsymbol{\sigma},L}) \big\|_{L^\infty(D,\mathbb{R}^{3+3L})} \quad = \quad \max \big\{ \|\vartheta\|_{L^\infty(D)}, \ldots, \|\omega_{\boldsymbol{\sigma},L}\|_{L^\infty(D)} \big\}, \tag{6.30}$$

$(\vartheta, \ldots, \omega_{\boldsymbol{\sigma},L})^\top \in L^\infty(D, \mathbb{R}^{3+3L})$.

Since by Lemma 60 it holds $P_1, P_2 \in \mathcal{L}(X)$ for $(\vartheta, \ldots, \omega_{\boldsymbol{\sigma},L}) \in L^\infty(D, \mathbb{R}^{3+3L})$, we can define the map

$$\begin{aligned} \Gamma : \qquad \mathcal{P} \quad &\to \quad \mathcal{L}(X)^2, \\ \big(\vartheta, \ldots, \omega_{\boldsymbol{\sigma},L}\big) \quad &\mapsto \quad (P_1, P_2), \end{aligned} \tag{6.31}$$

and extend it to a linear map $D\Gamma : L^\infty(D, \mathbb{R}^{3+3L}) \to \mathcal{L}(X)^2$ by using the same formal definition.

The set $\Gamma(\mathcal{P})$ will take the role from U in Assumption 82, and the set $\mathcal{T}\,\Gamma(\mathcal{P}) \coloneqq D\Gamma L^\infty(D, \mathbb{R}^{3+3L})$ will take the role from $\mathcal{T}U$.

For $(P_1, P_2) \in \Gamma(\mathcal{P})$, Lemma 61 states that P_1 is self-adjoint, monotone and boundedly invertible with respect to $(.\,,.)_X$. So we again have the scalar product

$$(w_1, w_2)_{E,P} \;\coloneqq\; (P_1^{-1} w_1, w_2)_X\,, \qquad\qquad w_1, w_2 \in X\,,$$

which we simply denoted by $(.\,,.)_E$ in (5.32).

As in Assumption 82, we endow $\mathcal{D}(Q)$ with the graph norms

$$\|w\|_{Q,X} \;=\; \|Qw\|_X + \|w\|_X\,,$$

and

$$\|w\|_{\beta(P),E,\tilde{P}} \;=\; \|\beta(P)w\|_{E,\tilde{P}} + \|w\|_{E,\tilde{P}}\,,$$

$w \in \mathcal{D}(Q)$, where $P, \tilde{P} \in \Gamma(\mathcal{P})$ are arbitrary. In further analogy to Assumption 82 the space $\mathcal{T}\,\Gamma(\mathcal{P})$ is normed by

$$\|(P_1, P_2)\|_{\mathcal{T}\,\Gamma(\mathcal{P}),X} \quad\coloneqq\quad \max\left\{\|P_1\|_{\mathcal{L}(X,\|.\|_X)}\,,\, \|P_2\|_{\mathcal{L}(X,\|.\|_X)}\right\}, \tag{6.32}$$

and

$$\|(P_1, P_2)\|_{\mathcal{T}\,\Gamma(\mathcal{P}),E,\tilde{P}} \quad\coloneqq\quad \max\left\{\|P_1\|_{\mathcal{L}(X,\|.\|_{E,\tilde{P}})}\,,\, \|P_2\|_{\mathcal{L}(X,\|.\|_{E,\tilde{P}})}\right\},$$

$(P_1, P_2) \in \mathcal{T}\,\Gamma(\mathcal{P})$, where $\tilde{P} \in \Gamma(\mathcal{P})$ is arbitrary.

Due to Corollary 71, $\beta(P)$ is maximal monotone with respect to $(.\,,.)_{E,P}$ for every $P \in \Gamma(\mathcal{P})$. □

It remains to assure that $\Gamma(\mathcal{P})$ is open in $\big(\mathcal{T}\,\Gamma(\mathcal{P}), \|.\|_{\mathcal{T}\,\Gamma(\mathcal{P}),X}\big)$. This will be the statement of Corollary 103. To prove it, we need the subsequent two lemmas.

Lemma 101. *The material parameter set $\mathcal{P}$ is an open subset of the normed space $\big(L^\infty(D, \mathbb{R}^{3+3L}), \|.\|_{L^\infty(D,\mathbb{R}^{3+3L})}\big)$.*

Proof. Let $\hat{p} = (\hat{p}_1, \ldots, \hat{p}_{3+3L}) \in \mathcal{P}$. Then there are $c_i > 0$ with $\hat{p}_i(x) \geq c_i$ for almost all $x \in D$ and all $i = 1, \ldots, 3+3L$. Let $\delta \coloneqq \min\{c_1, \ldots, c_{3+3L}\}/2 > 0$. Then for every $p \coloneqq (p_1, \ldots, p_{3+3L}) \in L^\infty(D, \mathbb{R}^{3+3L})$ with $\|\hat{p} - p\|_{L^\infty(D,\mathbb{R}^{3+3L})} < \delta$, it holds

$$p_i(x) \;=\; \hat{p}_i(x) \;-\; \big(\hat{p}_i(x) - p_i(x)\big) \;>\; 2\delta \;-\; \delta \;=\; \delta\,, \qquad \text{f.a.a. } x \in D\,,$$

$i = 1, \ldots, 3+3L$. So $p \in \mathcal{P}$. □

Lemma 102. *The map $D\Gamma$ is an isometry onto its image with respect to the norms $\|.\|_{L^\infty(D,\mathbb{R}^{3+3L})}$ like in* (6.30) *and $\|.\|_{\mathcal{T}\,\Gamma(\mathcal{P}),X}$ like in* (6.32).

Proof. Let $(\vartheta,\dots,\omega_{\boldsymbol{\sigma},L}) \in L^\infty(D,\mathbb{R}^{3+3L})$ and $(P_1,P_2) := D\Gamma(\vartheta,\dots,\omega_{\boldsymbol{\sigma},L})$. Then for every $l = 1,\dots,L$, the bounded linear operator $L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}}) \to L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})$, $\boldsymbol{\sigma}_{M,l} \mapsto \omega_{\boldsymbol{\sigma},l}\boldsymbol{\sigma}_{M,l}$ has norm $\|\omega_{\boldsymbol{\sigma},l}\|_{L^\infty(D)}$. After applying Lemma 50 (d) and (e) and rewriting $\omega_{\boldsymbol{\sigma},l}\boldsymbol{\sigma}_{M,l}$ in the form $\tilde{C}(\omega_{\boldsymbol{\sigma},l},\omega_{\boldsymbol{\sigma},l})\boldsymbol{\sigma}_{M,l}$, this is stated in (5.16). With the same methods used in the proof of Lemma 51, we also prove that the linear operator $L^2(D,\mathbb{R}^3) \to L^2(D,\mathbb{R}^3)$, $\mathbf{v} \mapsto \vartheta\mathbf{v}$ has norm $\|\vartheta\|_{L^\infty(D)}$. Furthermore, with Lemma 51, it follows

$$
\begin{aligned}
&\|P_1\|_{\mathcal{L}(X,\|\cdot\|_X)} \\
&\quad = \max\Big\{\|\vartheta\|_{L^\infty(D)},\ \big\|C(\mu_H,\kappa_H)\big\|_{\mathcal{L}(L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}}))}, \\
&\qquad\qquad \big\|C(\mu_{M,1}\,,\,\kappa_{M,1})\big\|_{\mathcal{L}(L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}}))},\ \dots,\ \big\|C(\mu_{M,L}\,,\,\kappa_{M,L})\big\|_{\mathcal{L}(L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}}))}\Big\} \\
&\quad = \max\Big\{\|\vartheta\|_{L^\infty(D)},\ \|\mu_H\|_{L^\infty(D)},\ \|\kappa_H\|_{L^\infty(D)},\ \|\mu_{M,1}\|_{L^\infty(D)},\ \dots,\ \|\kappa_{M,L}\|_{L^\infty(D)}\Big\}.
\end{aligned}
$$

In the same way we find

$$
\|P_2\|_{\mathcal{L}(X,\|\cdot\|_X)} \quad = \quad \max\big\{0,\ 0,\ \|\omega_{\boldsymbol{\sigma},1}\|_{L^\infty(D)},\ \dots,\ \|\omega_{\boldsymbol{\sigma},L}\|_{L^\infty(D)}\big\}.
$$

So from the form of $\|.\|_{L^\infty(D,\mathbb{R}^{3+3L})}$ and $\|.\|_{\mathcal{T}\,\Gamma(\mathcal{P}),X}$ given in (6.30) and (6.32), the statement follows. □

Corollary 103. *The set $\Gamma(\mathcal{P})$ is a (relatively) open subset of $\big(\mathcal{T}\,\Gamma(\mathcal{P}),\|.\|_{\mathcal{T}\,\Gamma(\mathcal{P}),X}\big)$.*

Proof. This is a consequence of Lemma 101 and Lemma 102. □

Remark 104. *Lemma 102 allows us to identify the material parameters in $\mathcal{P}$ with the corresponding pairs of bounded linear operators in $\Gamma(\mathcal{P})$, which in turn corresponds to U in Assumption 82. In the same way, the tangential vectors in $L^\infty(D,\mathbb{R}^{3+3L})$ can be identified with the corresponding pairs of bounded linear operators in $\mathcal{T}\,\Gamma(\mathcal{P})$, which corresponds to TU in Assumption 82.* □

As an additional result, the following lemma derives the connection between the inverse P_1^{-1} of any P_1 and the corresponding parameters $p \in \mathcal{P}$.

Lemma 105. *For $P = (P_1,P_2) = \Gamma(p)$ with $p = (\vartheta,\dots,\omega_{\boldsymbol{\sigma},L}) \in \mathcal{P}$ it holds*

$$
\begin{aligned}
&(P_1^{-1},P_2) \\
&\quad = \quad \Gamma\Big(\frac{1}{\vartheta},\ \frac{1}{\mu_H},\ \frac{1}{\kappa_H},\ \frac{1}{\mu_{M,1}},\ \dots,\ \frac{1}{\mu_{M,L}},\ \frac{1}{\kappa_{M,1}},\ \dots,\ \frac{1}{\kappa_{M,L}},\ \omega_{\boldsymbol{\sigma},1},\ \dots,\ \omega_{\boldsymbol{\sigma},L}\Big).
\end{aligned}
$$

Proof. This is proven by using Lemma 50(g). $\square$

Now that Assumption 82 is completely assured, we prepare the formulation of the central Definition 108 of this section.

We begin by recalling the abstract parameter-to-solution-map

$$\begin{aligned} G\ :\ \ \big(\Gamma(\mathcal{P}), \|.\|_{\mathcal{T}\Gamma(\mathcal{P}),X}\big)\ \times\ \big(\mathcal{D}(Q), \|.\|_{Q,X}\big)\ \times\ W^{1,1}\big((0,t_1),(X,\|.\|_X)\big) \\ \to\ \ C^1\Big([0,t_1],(X,\|.\|_X)\Big) \cap C\Big([0,t_1],\big(\mathcal{D}(Q),\|.\|_{Q,X}\big)\Big)\,, \\ (P,w_0,f)\ \ \mapsto\ \ w\,, \end{aligned}$$

defined in (6.4), where $(P,w_0,f) = \big((P_1,P_2),\,(\mathbf{v}^{(0)},\boldsymbol{\sigma}_H^{(0)},\boldsymbol{\sigma}_M^{(0)})^\top,\,(\vartheta\mathbf{f},\mathbf{g},\mathbf{0})^\top\big)$ with P_1, P_2 as in (6.29), and $w = (\mathbf{v},\boldsymbol{\sigma}_H,\boldsymbol{\sigma}_M)^\top$ is the solution of (6.27) in this application.

After having assured the conditions in Assumption 82, Theorem 90 states that G is differentiable when interpreted with the bigger codomain $C\big([0,t_1],(X,\|.\|_X)\big)$. Along with the material parameters encoded in the pair of operators (P_1,P_2) and the scaled external force density $\vartheta\mathbf{f}$ and external stress rate $\mathbf{g}$, it relates the initial values $\mathbf{v}^{(0)}$, $\boldsymbol{\sigma}_H^{(0)}$, $\boldsymbol{\sigma}_M^{(0)}$ of the transformed variables to the resulting wave field $(\mathbf{v},\boldsymbol{\sigma}_H,\boldsymbol{\sigma}_M)^\top$, which is also expressed in the new variables. To get this relation in terms of the original variables $\mathbf{v}$, $\boldsymbol{\sigma}$ and $\boldsymbol{\eta}$, we concatenate G with the variable transformation T.

Here, a technical difficulty arises, since the variable transformation T itself depends on the material parameters $\vartheta,\ldots,\omega_{\boldsymbol{\sigma},L}$, which we differentiate for in particular. On an abstract level, this problem was studied in section 6.1.2. In the sequel we therefore show, that the preliminaries listed in Assumption 93 are satisfied for the viscoelastic wave equation.

Lemma 106. *For the variable transformation T^{-1} in* (6.28) *it holds*

$$T^{-1}\ =\ \Phi\ -\ P_2$$

with P_2 as in (6.29) *and*

$$\Phi:\quad X\to X\,,\qquad \begin{pmatrix}\mathbf{v}\\ \boldsymbol{\sigma}_H\\ \boldsymbol{\sigma}_{M,1}\\ \vdots\\ \boldsymbol{\sigma}_{M,L}\end{pmatrix} \mapsto \begin{pmatrix}\mathbf{v}\\ \boldsymbol{\sigma}_H+\sum_{l=1}^L\boldsymbol{\sigma}_{M,l}\\ \mathbf{0}\\ \vdots\\ \mathbf{0}\end{pmatrix}.$$

Therefore we consider T^{-1}, $T\in\mathcal{L}(X,\|.\|_X)$ as functions of $(P_1,P_2)\in\Gamma(\mathcal{P})$ and from now on write $T(P)^{-1}$ and $T(P)$, respectively, in accordance with the notation in Assumption 93.

$P \mapsto T(P)$ and $P \mapsto T(P)^{-1}$ are Fréchet-differentiable, and their derivatives at a point $\hat{P} := \Gamma(\hat{p})$ with $\hat{p} = (\hat{\vartheta}, \dots, \hat{\omega}_{\boldsymbol{\sigma},L}) \in \mathcal{P}$ in a direction $P = (P_1, P_2) := D\Gamma p$ with $p = (\vartheta, \dots, \omega_{\boldsymbol{\sigma},L}) \in L^\infty(D, \mathbb{R}^{3+3L})$ are given by

$$\big[DT(\hat{P})P\big] \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma} \\ \boldsymbol{\eta} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ -\sum_{l=1}^{L} \frac{\omega_{\boldsymbol{\sigma},l}}{\hat{\omega}^2_{\boldsymbol{\sigma},l}} \boldsymbol{\eta}_l \\ \frac{\omega_{\boldsymbol{\sigma},1}}{\hat{\omega}^2_{\boldsymbol{\sigma},1}} \boldsymbol{\eta}_1 \\ \vdots \\ \frac{\omega_{\boldsymbol{\sigma},L}}{\hat{\omega}^2_{\boldsymbol{\sigma},L}} \boldsymbol{\eta}_L \end{pmatrix}, \qquad DT(\hat{P})^{-1}P = -P_2 \,.$$

Proof. This can be verified via straight forward calculations. □

Furthermore,

$$T(P)^{-1}\mathcal{D}(Q) = T(P)^{-1}\mathcal{D}\big(\beta(P)\big) = T(P)^{-1}T(P)\mathcal{D}(A) = \mathcal{D}(A)$$

due to the definition $\mathcal{D}(B) = T\mathcal{D}(A)$ of $\mathcal{D}(B)$ in (5.27) with $\mathcal{D}(A)$ as in (6.26), which is independent of $P \in \Gamma(\mathcal{P})$. Also

$$\big[DT(\hat{P})^{-1}P\big]\mathcal{D}(Q) = -P_2\mathcal{D}(Q) \subseteq \{\mathbf{0}\} \times \{\mathbf{0}\} \times L^2(D, \mathbb{R}^{3\times3}_{\mathrm{sym}})^L \subseteq \mathcal{D}(A)\,,$$

$\hat{P} \in \Gamma(\mathcal{P})$, $P \in \mathcal{T}\Gamma(\mathcal{P})$.

Finally, Assumption 93 requires an operator norm $\|.\|_{\tilde{Q},X}$ on $\mathcal{D}(A)$ corresponding to an operator $\tilde{Q}$ with $\mathcal{D}(\tilde{Q}) = \mathcal{D}(A)$, which is independent of the material parameters and locally equivalent to the material parameter dependent operator norm $\|.\|_{A,X} = \|A\,.\|_X + \|.\|_X$. This last premise is provided by the following lemma.

Lemma 107. *To indicate the dependence of the operator A in* (6.25) *on the parameters $p = (\vartheta, \dots, \omega_{\boldsymbol{\sigma},L}) \in \mathcal{P}$, we denote A by $\alpha(p)$ in this lemma. We furthermore introduce the operator $\tilde{Q} : \mathcal{D}(A) \to X$ with*

$$\tilde{Q} \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma} \\ \boldsymbol{\eta} \end{pmatrix} := \begin{pmatrix} \operatorname{div} \boldsymbol{\sigma} \\ \varepsilon(\mathbf{v}) \\ -\varepsilon(\mathbf{v}) \\ \vdots \\ -\varepsilon(\mathbf{v}) \end{pmatrix}.$$

Since the domain of definition $\mathcal{D}(A) = \mathcal{D}\big(\alpha(p)\big)$ does not depend on $p \in \mathcal{P}$, we use the definition

$$\mathcal{D}(\tilde{Q}) := \mathcal{D}(A)$$

throughout this section. For the graph norms $\|.\|_{\alpha(p),X} := \|\alpha(p)\,.\,\|_X + \|.\|_X$ *with* $p \in \mathcal{P}$ *and* $\|.\|_{\tilde{Q},X} := \|\tilde{Q}\,.\,\|_X + \|.\|_X$*, the following statement holds true.*

For every fixed $\hat{p} \in \mathcal{P}$*, there is a neighborhood* $\Omega \subseteq \mathcal{P}$ *of* $\hat{p}$ *and a constant* $c > 0$*, such that*

$$\frac{1}{c}\,\|u\|_{\tilde{Q},X} \;\le\; \|u\|_{\alpha(p),X} \;\le\; c\,\|u\|_{\tilde{Q},X}\,, \qquad u \in \mathcal{D}(\tilde{Q})\,, \qquad p \in \Omega\,.$$

Proof. It is $A = -\tilde{P}_1\tilde{Q} + \tilde{P}_2$, where $\tilde{P}_1, \tilde{P}_2 : X \to X$ with

$$\tilde{P}_1 \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma} \\ \boldsymbol{\eta} \end{pmatrix} := \begin{pmatrix} \vartheta\mathbf{v} \\ C\Big(\mu_H + \sum_{l=1}^L \mu_{M,l}\,,\, \kappa_H + \sum_{l=1}^L \kappa_{M,l}\Big)\boldsymbol{\sigma} \\ \omega_{\boldsymbol{\sigma},1} C(\mu_{M,1}, \kappa_{M,1})\boldsymbol{\eta}_1 \\ \vdots \\ \omega_{\boldsymbol{\sigma},L} C(\mu_{M,L}, \kappa_{M,L})\boldsymbol{\eta}_L \end{pmatrix},$$

$$\tilde{P}_2 \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma} \\ \boldsymbol{\eta} \end{pmatrix} := \begin{pmatrix} \mathbf{0} \\ -\sum_{l=1}^L \boldsymbol{\eta}_l \\ \omega_{\boldsymbol{\sigma},1}\boldsymbol{\eta}_1 \\ \vdots \\ \omega_{\boldsymbol{\sigma},L}\boldsymbol{\eta}_l \end{pmatrix}.$$

Now let $\hat{p} = (\hat{p}_1, \dots, \hat{p}_{3+3L}) \in \mathcal{P}$. There are $c_1, \dots, c_{3+3L} > 0$ with $\hat{p}_i \ge c_i$, $i = 1, \dots, 3+3L$. We define $\delta := \min\{c_1, \dots, c_{3+3L}\}/2$, $d_1 := \|\hat{p}\|_{L^\infty(D,\mathbb{R}^{3+3L})} - \delta$ and $d_2 := \|\hat{p}\|_{L^\infty(D,\mathbb{R}^{3+3L})} + \delta$.

For $(\vartheta, \dots, \omega_{\boldsymbol{\sigma},L}) \in B(\hat{p}, \delta) =: \Omega$, which is the open ball round $\hat{p}$ with radius δ, it is $\|\vartheta\|_{L^\infty(D)} \le d_2$, $\|1/\vartheta\|_{L^\infty(D)} \le 1/d_1$, $\|\mu_H + \sum_{l=1}^L \mu_{M,l}\|_{L^\infty(D)} \le (1+L)d_2$, $\|1/(\mu_H + \sum_{l=1}^L \mu_{M,l})\|_{L^\infty(D)} \le 1/((1+L)d_1)$, $\|\kappa_H + \sum_{l=1}^L \kappa_{M,l}\|_{L^\infty(D)} \le (1+L)d_2$, $\|1/(\kappa_H + \sum_{l=1}^L \kappa_{M,l})\|_{L^\infty(D)} \le 1/((1+L)d_1)$, $\|\omega_{\boldsymbol{\sigma},l}\mu_{M,l}\|_{L^\infty(D)} \le d_2^2$, $\|1/(\omega_{\boldsymbol{\sigma},l}\mu_{M,l})\|_{L^\infty(D)} \le 1/d_1^2$, $\|\omega_{\boldsymbol{\sigma},l}\kappa_{M,l}\|_{L^\infty(D)} \le d_2^2$, $\|1/(\omega_{\boldsymbol{\sigma},l}\kappa_{M,l})\|_{L^\infty(D)} \le 1/d_1^2$, $\|\omega_{\boldsymbol{\sigma},l}\|_{L^\infty(D)} \le d_2$, $l = 1, \dots, L$.

With $d := \max\big\{(1+L)d_2, d_2^2, 1/d_1, 1/d_1^2, \sqrt{L + d_2^2}\,\big\}$ we find that

$$\|\tilde{P}_1\|_{\mathcal{L}(X,\|\cdot\|_X)}\,,\; \|\tilde{P}_2\|_{\mathcal{L}(X,\|\cdot\|_X)}\,,\; \|\tilde{P}_1^{-1}\|_{\mathcal{L}(X,\|\cdot\|_X)}\,,\; \|\mathrm{Id}\|_{\mathcal{L}(X,\|\cdot\|_X)} \quad\le\quad d\,.$$

So we can apply estimate (4.16), where we plug in $\tilde{P}_1$, $\tilde{P}_2$ and $\tilde{Q}$ from this lemma for P_1, P_2 and Q in Lemma 36, respectively, and the identity operator $\mathrm{Id} \in \mathcal{L}(X)$ for $\tilde{P}_1$ in Lemma 36. ◻

Now we turn to the central definition of this section.

Definition 108. *As the **parameter-to-solution-map***

$$\begin{aligned} F: \quad & \big(\mathcal{P}, \|.\|_{L^\infty(D,\mathbb{R}^{3+3L})}\big) \times \big(\mathcal{D}(\tilde{Q}), \|.\|_{\tilde{Q},X}\big) \times W^{1,1}\big((0,t_1),\, L^2(D, \mathbb{R}^3 \times \mathbb{R}^{3\times 3}_{\mathrm{sym}})\big) \\ & \to \quad C\big([0,t_1]\,,\, L^2(D, \mathbb{R}^3 \times \mathbb{R}^{3\times 3}_{\mathrm{sym}})\big) \end{aligned}$$

we denote the map, which maps every tuple of parameters

$$\Big((\vartheta, \ldots, \omega_{\boldsymbol{\sigma},L}),\, \big(\mathbf{v}^{(0)}, \boldsymbol{\sigma}^{(0)}, \boldsymbol{\eta}^{(0)}\big)^\top,\, (\mathbf{f}, \mathbf{g})^\top\Big)$$

to the pair $(\mathbf{v}, \boldsymbol{\sigma})^\top$, where $(\mathbf{v}, \boldsymbol{\sigma}, \boldsymbol{\eta})^\top$ is the unique solution of the initial-boundary value problem

$$\begin{aligned} \mathbf{v}'(t) \quad &= \quad \vartheta \operatorname{div} \boldsymbol{\sigma}(t) \;+\; \vartheta\, \mathbf{f}(t)\,, \\ \boldsymbol{\sigma}'(t) \quad &= \quad C\Big(\mu_H + \sum_{l=1}^{L} \mu_{M,l}\,,\; \kappa_H + \sum_{l=1}^{L} \kappa_{M,l}\Big)\, \varepsilon\big(\mathbf{v}(t)\big) \;+\; \sum_{l=1}^{L} \boldsymbol{\eta}_l(t) \;+\; \mathbf{g}(t)\,, \\ \boldsymbol{\eta}_l'(t) \quad &= \quad -\omega_{\boldsymbol{\sigma},l} C\big(\mu_{M,l}\,,\, \kappa_{M,l}\big)\, \varepsilon\big(\mathbf{v}(t)\big) \;-\; \omega_{\boldsymbol{\sigma},l} \boldsymbol{\eta}_l(t)\,, \qquad l = 1, \ldots, L\,, \end{aligned}$$

$$\mathbf{v}(0) \;=\; \mathbf{v}^{(0)}\,, \qquad \boldsymbol{\sigma}(0) \;=\; \boldsymbol{\sigma}^{(0)}\,, \qquad \boldsymbol{\eta}(0) \;=\; \boldsymbol{\eta}^{(0)}\,,$$
$$\mathbf{v}(t)\big|_{\partial D_D} \;=\; \mathbf{0}\,, \qquad \mathbf{n}^\top \boldsymbol{\sigma}(t)\big|_{\partial D_N} \;=\; \mathbf{0}\,,$$

$t \in [0, t_1]$, in the function space $C^1\big([0,t_1], X\big) \cap C\big([0,t_1], \mathcal{D}(\tilde{Q})\big)$. □

To express F in terms of the back-transformed abstract parameter-to-solution-map H introduced in Definition 95, we define two additional functions.

Notation 109. By

$$\begin{aligned} \pi_2 \;:\quad C\big([0,t_1], (X, \|.\|_X)\big) \quad &\to \quad C\big([0,t_1]\,,\, L^2(D, \mathbb{R}^3 \times \mathbb{R}^{3\times 3}_{\mathrm{sym}})\big)\,, \\ (\mathbf{v}, \boldsymbol{\sigma}, \boldsymbol{\eta})^\top \quad &\mapsto \quad (\mathbf{v}, \boldsymbol{\sigma})^\top \end{aligned}$$

we denote the canonical projection onto the first two components. Furthermore, we introduce the map

$$\begin{aligned} \gamma: \quad & \big(\mathcal{P}, \|.\|_{L^\infty(D,\mathbb{R}^{3+3L})}\big) \times \big(\mathcal{D}(\tilde{Q}), \|.\|_{\tilde{Q},X}\big) \times W^{1,1}\big((0,t_1)\,,\, L^2(D, \mathbb{R}^3 \times \mathbb{R}^{3\times 3}_{\mathrm{sym}})\big) \\ & \to \quad \big(\Gamma(\mathcal{P}), \|.\|_{\mathcal{T}\,\Gamma(\mathcal{P}),X}\big) \times \big(\mathcal{D}(\tilde{Q}), \|.\|_{\tilde{Q},X}\big) \times W^{1,1}\big((0,t_1)\,,\, (X, \|.\|_X)\big)\,, \\ & \Big((\vartheta, \ldots, \omega_{\boldsymbol{\sigma},L}),\, \big(\mathbf{v}^{(0)}, \boldsymbol{\sigma}^{(0)}, \boldsymbol{\eta}^{(0)}\big)^\top,\, (\mathbf{f}, \mathbf{g})^\top\Big) \\ & \mapsto \quad \Big(\Gamma(\vartheta, \ldots, \omega_{\boldsymbol{\sigma},L}),\, \big(\mathbf{v}^{(0)}, \boldsymbol{\sigma}^{(0)}, \boldsymbol{\eta}^{(0)}\big)^\top,\, (\vartheta\mathbf{f}, \mathbf{g}, \mathbf{0})^\top\Big)\,. \end{aligned} \tag{6.33}$$

□

Lemma 110. *It is*

$$F = \pi_2 \circ H \circ \gamma \,,$$

where H denotes the map formally defined in Definition 95 as the back-transformed abstract parameter-to-solution-map.

Proof. This follows directly from the definition of the four functions involved, together with Theorem 65 about the variable transformation. □

In the remainder of this section we will prove the Fréchet-differentiability of F and derive explicit forms of its derivative in terms of the original variables $\mathbf{v}$, $\boldsymbol{\sigma}$, $\boldsymbol{\eta}$ as well as the transformed ones $\mathbf{v}$, $\boldsymbol{\sigma}_H$, $\boldsymbol{\sigma}_M$. These results are mainly based on Lemma 98 and Theorem 99, which in turn can be seen as corollaries of Theorem 90 on the differentiability of the abstract parameter-to-solution-map G.

Lemma 111. *The map γ from (6.33) is Fréchet-differentiable and the Fréchet-derivative at any point $\hat{p} = \big((\hat{\vartheta}, \dots, \hat{\omega}_{\boldsymbol{\sigma},L}),\, (\hat{\mathbf{v}}^{(0)}, \hat{\boldsymbol{\sigma}}^{(0)}, \hat{\boldsymbol{\eta}}^{(0)})^\top,\, (\hat{\mathbf{f}}, \hat{\mathbf{g}})^\top\big)$ of its domain is given by the bounded linear map*

$$\begin{aligned} D\gamma(\hat{p}) : \quad & L^\infty(D, \mathbb{R}^{3+3L}) \times \big(\mathcal{D}(\tilde{Q}), \|.\|_{\tilde{Q},X}\big) \times W^{1,1}\Big((0,t_1)\,,\, L^2(D, \mathbb{R}^3 \times \mathbb{R}^{3\times 3}_{\mathrm{sym}})\Big) \\ & \to \big(\mathcal{T}\,\Gamma(\mathcal{P}), \|.\|_{\mathcal{T}\,\Gamma(\mathcal{P}),X}\big) \times \big(\mathcal{D}(\tilde{Q}), \|.\|_{\tilde{Q},X}\big) \times W^{1,1}\big((0,t_1)\,,\,(X, \|.\|_X)\big)\,, \\ & \Big((\vartheta, \dots, \omega_{\boldsymbol{\sigma},L}),\, \big(\mathbf{v}^{(0)}, \boldsymbol{\sigma}^{(0)}, \boldsymbol{\eta}^{(0)}\big)^\top,\, (\mathbf{f}, \mathbf{g})^\top\Big) \\ & \mapsto \Big(D\Gamma(\vartheta, \dots, \omega_{\boldsymbol{\sigma},L}),\, \big(\mathbf{v}^{(0)}, \boldsymbol{\sigma}^{(0)}, \boldsymbol{\eta}^{(0)}\big)^\top,\, (\vartheta\hat{\mathbf{f}} + \hat{\vartheta}\mathbf{f}, \mathbf{g}, \mathbf{0})^\top\Big)\,. \end{aligned}$$

Furthermore, it is

$$\pi_2\, T\big(\Gamma(p)\big)^{-1} \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma}_H \\ \boldsymbol{\sigma}_M \end{pmatrix} = \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma}_H + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l} \end{pmatrix}\,,$$

$(\mathbf{v}, \boldsymbol{\sigma}_H, \boldsymbol{\sigma}_M)^\top \in C\big([0,t_1],\, X\big),\ p \in \mathcal{P}.$

Proof. Since the map $D\Gamma$ is linear and is the extension of Γ onto the whole of $L^\infty(D, \mathbb{R}^{3+3L})$, it is $D\Gamma p$ the Fréchet-derivative of Γ at any point $\hat{p} \in \mathcal{P}$ in direction $p \in L^\infty(D, \mathbb{R}^{3+3L})$.

That the map

$$\begin{aligned} L^\infty_+(D) \times W^{1,1}\big((0,t_1)\,,\, L^2(D, \mathbb{R}^3)\big) \quad &\to \quad W^{1,1}\big((0,t_1)\,,\, L^2(D, \mathbb{R}^3)\big) \\ (\vartheta, \mathbf{f}) \quad &\mapsto \quad \vartheta\mathbf{f} \end{aligned}$$

has derivative $\vartheta\hat{\mathbf{f}} + \hat{\vartheta}\mathbf{f}$ at any point $(\hat{\vartheta}, \hat{\mathbf{f}})$ in any direction $(\vartheta, \mathbf{f})$, follows from Lemma 96. Now Lemma 97 yields the first statement of this lemma.

The second one is verified by a direct calculation. □

Theorem 112. *The parameter-to-solution-map F is Fréchet-differentiable, and the Fréchet-derivative*

$$\begin{aligned} DF : &\big(\mathcal{P}, \|.\|_{L^\infty(D,\mathbb{R}^{3+3L})}\big) \times \big(\mathcal{D}(\tilde{Q}), \|.\|_{\tilde{Q},X}\big) \times W^{1,1}\big((0,t_1),\, L^2(D,\mathbb{R}^3 \times \mathbb{R}^{3\times3}_{\mathrm{sym}})\big) \\ &\to \mathcal{L}\Big(L^\infty(D,\mathbb{R}^{3+3L}) \times \big(\mathcal{D}(\tilde{Q}), \|.\|_{\tilde{Q},X}\big) \times W^{1,1}\big((0,t_1),\, L^2(D,\mathbb{R}^3 \times \mathbb{R}^{3\times3}_{\mathrm{sym}})\big)\,, \\ &\qquad C\big([0,t_1]\,,\, L^2(D,\mathbb{R}^3 \times \mathbb{R}^{3\times3}_{\mathrm{sym}})\big)\Big) \end{aligned}$$

at a point $\big((\hat{\vartheta},\, \ldots,\, \hat{\omega}_{\boldsymbol{\sigma},L}),\, (\hat{\mathbf{v}}^{(0)},\, \hat{\boldsymbol{\sigma}}^{(0)},\, \hat{\boldsymbol{\eta}}^{(0)})^\top,\, (\hat{\mathbf{f}},\, \hat{\mathbf{g}})^\top\big) \in \mathcal{P} \times \mathcal{D}(\tilde{Q}) \times W^{1,1}\big((0,t_1),$ $L^2(D,\mathbb{R}^3 \times \mathbb{R}^{3\times3}_{\mathrm{sym}})\big)$ *in a direction* $\big((\vartheta,\, \ldots,\, \omega_{\boldsymbol{\sigma},L}),\, (\mathbf{v}^{(0)},\, \boldsymbol{\sigma}^{(0)},\, \boldsymbol{\eta}^{(0)})^\top,\, (\mathbf{f},\, \mathbf{g})^\top\big) \in$ $L^\infty(D,\mathbb{R}^{3+3L}) \times \mathcal{D}(\tilde{Q}) \times W^{1,1}\big((0,t_1),\, L^2(D,\mathbb{R}^3 \times \mathbb{R}^{3\times3}_{\mathrm{sym}})\big)$ *is given by the pair* $(\mathbf{v},\boldsymbol{\sigma})^\top$, *where* $(\mathbf{v},\boldsymbol{\sigma},\boldsymbol{\eta})^\top \in C\big([0,t_1],X\big)$ *is the mild solution of*

$$\begin{aligned} \mathbf{v}'(t) \;&=\; \hat{\vartheta}\,\mathrm{div}\,\boldsymbol{\sigma}(t) \;+\; \vartheta\,\mathrm{div}\,\hat{\boldsymbol{\sigma}}(t) \;+\; \hat{\vartheta}\,\mathbf{f}(t) \;+\; \vartheta\,\hat{\mathbf{f}}(t)\,, \\ \boldsymbol{\sigma}'(t) \;&=\; C\bigg(\hat{\mu}_H + \sum_{l=1}^{L} \hat{\mu}_{M,l}\;,\; \hat{\kappa}_H + \sum_{l=1}^{L} \hat{\kappa}_{M,l}\bigg)\, \varepsilon\big(\mathbf{v}(t)\big) \\ &\quad+\; C\bigg(\mu_H + \sum_{l=1}^{L} \mu_{M,l}\;,\; \kappa_H + \sum_{l=1}^{L} \kappa_{M,l}\bigg)\, \varepsilon\big(\hat{\mathbf{v}}(t)\big) \\ &\quad+\; \sum_{l=1}^{L} \boldsymbol{\eta}_l(t) \;+\; \mathbf{g}(t)\,, \\ \boldsymbol{\eta}_l'(t) \;&=\; -\,\hat{\omega}_{\boldsymbol{\sigma},l} C\big(\hat{\mu}_{M,l}\,,\, \hat{\kappa}_{M,l}\big)\, \varepsilon\big(\mathbf{v}(t)\big) \;-\; \hat{\omega}_{\boldsymbol{\sigma},l} C\big(\mu_{M,l}\,,\, \kappa_{M,l}\big)\, \varepsilon\big(\hat{\mathbf{v}}(t)\big) \\ &\quad-\; \omega_{\boldsymbol{\sigma},l} C\big(\hat{\mu}_{M,l}\,,\, \hat{\kappa}_{M,l}\big)\, \varepsilon\big(\hat{\mathbf{v}}(t)\big) \\ &\quad-\; \hat{\omega}_{\boldsymbol{\sigma},l}\boldsymbol{\eta}_l(t) \;-\; \omega_{\boldsymbol{\sigma},l}\hat{\boldsymbol{\eta}}_l(t)\,, \qquad\qquad l = 1,\ldots,L\,, \end{aligned}$$

$$\begin{gathered} \mathbf{v}(0) \;=\; \mathbf{v}^{(0)}\,, \qquad \boldsymbol{\sigma}(0) \;=\; \boldsymbol{\sigma}^{(0)}\,, \qquad \boldsymbol{\eta}(0) \;=\; \boldsymbol{\eta}^{(0)}\,, \\ \mathbf{v}(t)\big|_{\partial D_D} \;=\; \mathbf{0}\,, \qquad \mathbf{n}^\top \boldsymbol{\sigma}(t)\big|_{\partial D_N} \;=\; \mathbf{0}\,, \end{gathered}$$

$t \in [0,t_1]$, *and* $(\hat{\mathbf{v}},\hat{\boldsymbol{\sigma}},\hat{\boldsymbol{\eta}})^\top$ *is the classical solution of*

$$\begin{aligned} \hat{\mathbf{v}}'(t) \;&=\; \hat{\vartheta}\,\mathrm{div}\,\hat{\boldsymbol{\sigma}}(t) \;+\; \hat{\vartheta}\,\hat{\mathbf{f}}(t)\,, \\ \hat{\boldsymbol{\sigma}}'(t) \;&=\; C\bigg(\hat{\mu}_H + \sum_{l=1}^{L} \hat{\mu}_{M,l}\;,\; \hat{\kappa}_H + \sum_{l=1}^{L} \hat{\kappa}_{M,l}\bigg)\, \varepsilon\big(\hat{\mathbf{v}}(t)\big) \;+\; \sum_{l=1}^{L} \hat{\boldsymbol{\eta}}_l(t) \;+\; \hat{\mathbf{g}}(t)\,, \\ \hat{\boldsymbol{\eta}}_l'(t) \;&=\; -\hat{\omega}_{\boldsymbol{\sigma},l} C\big(\hat{\mu}_{M,l}\,,\, \hat{\kappa}_{M,l}\big)\, \varepsilon\big(\hat{\mathbf{v}}(t)\big) \;-\; \hat{\omega}_{\boldsymbol{\sigma},l}\hat{\boldsymbol{\eta}}_l(t)\,, \qquad\qquad l = 1,\ldots,L\,, \end{aligned}$$

$$\begin{gathered} \hat{\mathbf{v}}(0) \;=\; \hat{\mathbf{v}}^{(0)}\,, \qquad \hat{\boldsymbol{\sigma}}(0) \;=\; \hat{\boldsymbol{\sigma}}^{(0)}\,, \qquad \hat{\boldsymbol{\eta}}(0) \;=\; \hat{\boldsymbol{\eta}}^{(0)}\,, \\ \hat{\mathbf{v}}(t)\big|_{\partial D_D} \;=\; \mathbf{0}\,, \qquad \mathbf{n}^\top \hat{\boldsymbol{\sigma}}(t)\big|_{\partial D_N} \;=\; \mathbf{0}\,, \end{gathered}$$

$t \in [0,t_1]$.

Proof. This follows from Theorem 99, Lemma 111, the linearity of π_2, and the chain rule.

We only have to calculate the form of the operator called A in Theorem 99. With the notation $\hat{P} = \Gamma(\hat{\vartheta}, \ldots, \hat{\omega}_{\boldsymbol{\sigma},L})$, $P = (P_1, P_2) = D\Gamma(\vartheta, \ldots, \omega_{\boldsymbol{\sigma},L})$, $\hat{A} = T(\hat{P})^{-1}\beta(\hat{P})T(\hat{P})$, and the formula $DT(\hat{P})^{-1}P = -P_2$, we evaluate

$$A \quad = \quad -P_2\beta(\hat{P})T(\hat{P}) \; + \; \hat{A}P_2T(\hat{P}) \; + \; T(\hat{P})^{-1}\beta(P)T(\hat{P})\,.$$

For $\hat{u} = (\hat{\mathbf{v}}, \hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\eta}})^\top \in \mathcal{D}(\tilde{Q})$, a direct calculation shows

$$\begin{aligned}
-P_2\beta(\hat{P})T(\hat{P})\,\hat{u} \quad &= \quad -P_2 \begin{pmatrix} -\,\hat{\vartheta}\operatorname{div}\hat{\boldsymbol{\sigma}} \\ -\,C\big(\hat{\mu}_H, \hat{\kappa}_H\big)\,\varepsilon(\hat{\mathbf{v}}) \\ -\,C\big(\hat{\mu}_{M,1}, \hat{\kappa}_{M,1}\big)\,\varepsilon(\hat{\mathbf{v}}) - \hat{\boldsymbol{\eta}}_l \\ \vdots \\ -\,C\big(\hat{\mu}_{M,L}, \hat{\kappa}_{M,L}\big)\,\varepsilon(\hat{\mathbf{v}}) - \hat{\boldsymbol{\eta}}_L \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \omega_{\boldsymbol{\sigma},1}C\big(\hat{\mu}_{M,1}, \hat{\kappa}_{M,1}\big)\,\varepsilon(\hat{\mathbf{v}}) + \omega_{\boldsymbol{\sigma},1}\hat{\boldsymbol{\eta}}_l \\ \vdots \\ \omega_{\boldsymbol{\sigma},L}C\big(\hat{\mu}_{M,L}, \hat{\kappa}_{M,L}\big)\,\varepsilon(\hat{\mathbf{v}}) + \omega_{\boldsymbol{\sigma},L}\hat{\boldsymbol{\eta}}_L \end{pmatrix},
\end{aligned}$$

$$\hat{A}P_2T(\hat{P})\,\hat{u} \quad = \quad \hat{A}\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ -\frac{\omega_{\boldsymbol{\sigma},1}}{\hat{\omega}_{\boldsymbol{\sigma},1}}\hat{\boldsymbol{\eta}}_1 \\ \vdots \\ -\frac{\omega_{\boldsymbol{\sigma},L}}{\hat{\omega}_{\boldsymbol{\sigma},L}}\hat{\boldsymbol{\eta}}_L \end{pmatrix} \quad = \quad \begin{pmatrix} \mathbf{0} \\ \sum_{l=1}^L \frac{\omega_{\boldsymbol{\sigma},l}}{\hat{\omega}_{\boldsymbol{\sigma},l}}\hat{\boldsymbol{\eta}}_l \\ -\omega_{\boldsymbol{\sigma},1}\hat{\boldsymbol{\eta}}_1 \\ \vdots \\ -\omega_{\boldsymbol{\sigma},L}\hat{\boldsymbol{\eta}}_L \end{pmatrix},$$

$$\begin{aligned}
T(\hat{P})^{-1}\beta(P)T(\hat{P})\,\hat{u} \quad &= T(\hat{P})^{-1} \begin{pmatrix} -\,\vartheta\operatorname{div}\hat{\boldsymbol{\sigma}} \\ -\,C\big(\mu_H, \kappa_H\big)\,\varepsilon(\hat{\mathbf{v}}) \\ -\,C\big(\mu_{M,1}, \kappa_{M,1}\big)\,\varepsilon(\hat{\mathbf{v}}) - \frac{\omega_{\boldsymbol{\sigma},1}}{\hat{\omega}_{\boldsymbol{\sigma},1}}\hat{\boldsymbol{\eta}}_1 \\ \vdots \\ -\,C\big(\mu_{M,L}, \kappa_{M,L}\big)\,\varepsilon(\hat{\mathbf{v}}) - \frac{\omega_{\boldsymbol{\sigma},L}}{\hat{\omega}_{\boldsymbol{\sigma},L}}\hat{\boldsymbol{\eta}}_L \end{pmatrix} \\
= \quad &\begin{pmatrix} -\,\vartheta\operatorname{div}\hat{\boldsymbol{\sigma}} \\ -\,C\Big(\mu_H + \sum_{l=1}^L \mu_{M,l}\,,\, \kappa_H + \sum_{l=1}^L \kappa_{M,l}\Big)\,\varepsilon(\hat{\mathbf{v}}) - \sum_{l=1}^L \frac{\omega_{\boldsymbol{\sigma},l}}{\hat{\omega}_{\boldsymbol{\sigma},l}}\hat{\boldsymbol{\eta}}_l \\ \hat{\omega}_{\boldsymbol{\sigma},1}C\big(\mu_{M,1}, \kappa_{M,1}\big)\,\varepsilon(\hat{\mathbf{v}}) + \omega_{\boldsymbol{\sigma},1}\hat{\boldsymbol{\eta}}_1 \\ \vdots \\ \hat{\omega}_{\boldsymbol{\sigma},L}C\big(\mu_{M,L}, \kappa_{M,L}\big)\,\varepsilon(\hat{\mathbf{v}}) + \omega_{\boldsymbol{\sigma},L}\hat{\boldsymbol{\eta}}_L \end{pmatrix}.
\end{aligned}$$

So

$$A\,\hat{u} \quad = \quad \begin{pmatrix} -\vartheta\,\mathrm{div}\,\hat{\boldsymbol{\sigma}} \\ -\,C\Big(\mu_H + \sum_{l=1}^{L}\mu_{M,l}\,,\ \kappa_H + \sum_{l=1}^{L}\kappa_{M,l}\Big)\,\varepsilon(\hat{\mathbf{v}}) \\ \hat{\omega}_{\boldsymbol{\sigma},1}C\big(\mu_{M,1},\kappa_{M,1}\big)\,\varepsilon(\hat{\mathbf{v}}) + \omega_{\boldsymbol{\sigma},1}C\big(\hat{\mu}_{M,1},\hat{\kappa}_{M,1}\big)\,\varepsilon(\hat{\mathbf{v}}) + \omega_{\boldsymbol{\sigma},1}\hat{\boldsymbol{\eta}}_1 \\ \vdots \\ \hat{\omega}_{\boldsymbol{\sigma},L}C\big(\mu_{M,L},\kappa_{M,L}\big)\,\varepsilon(\hat{\mathbf{v}}) + \omega_{\boldsymbol{\sigma},L}C\big(\hat{\mu}_{M,L},\hat{\kappa}_{M,L}\big)\,\varepsilon(\hat{\mathbf{v}}) + \omega_{\boldsymbol{\sigma},L}\hat{\boldsymbol{\eta}}_L \end{pmatrix}.$$

□

How the derivative of the parameter-to-solution-map can be computed using the transformed system of equations (6.27) is shown in the next theorem.

Theorem 113. *The Fréchet-derivative of the parameter-to-solution-map F at some point $\big((\hat{\vartheta},\,\ldots,\,\hat{\omega}_{\boldsymbol{\sigma},L}),\,(\hat{\mathbf{v}}^{(0)},\,\hat{\boldsymbol{\sigma}}^{(0)},\,\hat{\boldsymbol{\eta}}^{(0)})^\top,\,(\hat{\mathbf{f}},\,\hat{\mathbf{g}})^\top\big) \in \mathcal{P}\times\mathcal{D}(\tilde{Q})\times W^{1,1}\big((0,t_1),$ $L^2(D,\mathbb{R}^3\times\mathbb{R}^{3\times3}_{\mathrm{sym}})\big)$ in some direction $\big((\vartheta,\,\ldots,\,\omega_{\boldsymbol{\sigma},L}),\,(\mathbf{v}^{(0)},\,\boldsymbol{\sigma}^{(0)},\,\boldsymbol{\eta}^{(0)})^\top,\,(\mathbf{f},\,\mathbf{g})^\top\big) \in$ $L^\infty(D,\mathbb{R}^{3+3L})\times\mathcal{D}(\tilde{Q})\times W^{1,1}\big((0,t_1),\,L^2(D,\mathbb{R}^3\times\mathbb{R}^{3\times3}_{\mathrm{sym}})\big)$ is given by the pair of functions $\big(\mathbf{v},\,\boldsymbol{\sigma}_H + \sum_{l=1}^{L}\boldsymbol{\sigma}_{M,l}\big)^\top$, where $(\mathbf{v},\boldsymbol{\sigma}_H,\boldsymbol{\sigma}_M)^\top \in C\big([0,t_1],X\big)$ is the mild solution of*

$$\begin{aligned}
\mathbf{v}'(t) \quad &= \quad \hat{\vartheta}\ \mathrm{div}\Big(\boldsymbol{\sigma}_H + \sum_{l=1}^{L}\boldsymbol{\sigma}_{M,l}\Big)(t) \ + \ \vartheta\ \mathrm{div}\Big(\hat{\boldsymbol{\sigma}}_H + \sum_{l=1}^{L}\hat{\boldsymbol{\sigma}}_{M,l}\Big)(t) \\
&\qquad + \ \vartheta\,\hat{\mathbf{f}}(t) \ + \ \hat{\vartheta}\,\mathbf{f}(t)\,, \\
\boldsymbol{\sigma}'_H(t) \quad &= \quad C(\hat{\mu}_H,\hat{\kappa}_H)\,\varepsilon\big(\mathbf{v}(t)\big) \ + \ C(\mu_H,\kappa_H)\,\varepsilon\big(\hat{\mathbf{v}}(t)\big) \ + \ \mathbf{g}(t)\,, \\
\boldsymbol{\sigma}'_{M,l}(t) \quad &= \quad C(\hat{\mu}_{M,l},\hat{\kappa}_{M,l})\,\varepsilon\big(\mathbf{v}(t)\big) \ + \ C(\mu_{M,l},\kappa_{M,l})\,\varepsilon\big(\hat{\mathbf{v}}(t)\big) \\
&\qquad - \ \hat{\omega}_{\boldsymbol{\sigma},l}\,\boldsymbol{\sigma}_{M,l}(t) \ - \ \omega_{\boldsymbol{\sigma},l}\,\hat{\boldsymbol{\sigma}}_{M,l}(t)\,, \qquad\qquad l = 1,\ldots,L\,,
\end{aligned}$$

$$\begin{aligned}
\mathbf{v}(0) \ = \ \mathbf{v}^{(0)}\,, \qquad\qquad \boldsymbol{\sigma}_H(0) \ &= \ \boldsymbol{\sigma}^{(0)} + \sum_{l=1}^{L}\Big(\frac{1}{\hat{\omega}_{\boldsymbol{\sigma},l}}\boldsymbol{\eta}_l^{(0)} - \frac{\omega_{\boldsymbol{\sigma},l}}{\hat{\omega}_{\boldsymbol{\sigma},l}^2}\hat{\boldsymbol{\eta}}_l^{(0)}\Big)\,, \\
\boldsymbol{\sigma}_{M,l}(0) \ &= \ -\frac{1}{\hat{\omega}_{\boldsymbol{\sigma},l}}\boldsymbol{\eta}_l^{(0)} + \frac{\omega_{\boldsymbol{\sigma},l}}{\hat{\omega}_{\boldsymbol{\sigma},l}^2}\hat{\boldsymbol{\eta}}_l^{(0)}\,, \qquad l = 1,\ldots,L\,,
\end{aligned}$$

$$\mathbf{v}(t)\big|_{\partial D_D} \ = \ \mathbf{0}\,, \qquad\qquad \mathbf{n}^\top\Big(\boldsymbol{\sigma}_H + \sum_{l=1}^{L}\boldsymbol{\sigma}_{M,l}\Big)(t)\Big|_{\partial D_N} \ = \ \mathbf{0}\,,$$

$t \in [0,t_1]$, and $(\hat{\mathbf{v}}, \hat{\boldsymbol{\sigma}}_H, \hat{\boldsymbol{\sigma}}_M)^\top$ is the classical solution of

$$\begin{aligned}
\hat{\mathbf{v}}'(t) &= \hat{\vartheta}\,\mathrm{div}\Big(\hat{\boldsymbol{\sigma}}_H + \sum_{l=1}^{L} \hat{\boldsymbol{\sigma}}_{M,l}\Big)(t) \;+\; \hat{\vartheta}\,\hat{\mathbf{f}}(t)\,,\\
\hat{\boldsymbol{\sigma}}'_H(t) &= C(\hat{\mu}_H, \hat{\kappa}_H)\,\varepsilon\big(\hat{\mathbf{v}}(t)\big) \;+\; \hat{\mathbf{g}}(t)\,,\\
\hat{\boldsymbol{\sigma}}'_{M,l}(t) &= C(\hat{\mu}_{M,l}, \hat{\kappa}_{M,l})\,\varepsilon\big(\hat{\mathbf{v}}(t)\big) \;-\; \hat{\omega}_{\boldsymbol{\sigma},l}\,\hat{\boldsymbol{\sigma}}_{M,l}(t)\,, \qquad l = 1,\dots,L\,,
\end{aligned}$$

$$\hat{\mathbf{v}}(0) \;=\; \hat{\mathbf{v}}^{(0)}\,, \qquad \hat{\boldsymbol{\sigma}}_H(0) \;=\; \hat{\boldsymbol{\sigma}}^{(0)} + \sum_{l=1}^{L} \frac{1}{\hat{\omega}_{\boldsymbol{\sigma},l}} \hat{\boldsymbol{\eta}}_l^{(0)}\,,$$

$$\hat{\boldsymbol{\sigma}}_{M,l}(0) \;=\; -\frac{1}{\hat{\omega}_{\boldsymbol{\sigma},l}} \hat{\boldsymbol{\eta}}_l^{(0)}\,, \qquad l = 1,\dots,L\,,$$

$$\hat{\mathbf{v}}(t)\big|_{\partial D_D} \;=\; \mathbf{0}\,, \qquad \mathbf{n}^\top \Big(\hat{\boldsymbol{\sigma}}_H + \sum_{l=1}^{L} \hat{\boldsymbol{\sigma}}_{M,l}\Big)(t)\Big|_{\partial D_N} \;=\; \mathbf{0}\,,$$

$t \in [0,t_1]$.

Proof. Since by Definition 95, which introduces the function H, it is

$$F\big(p, u_0, (\mathbf{f}, \mathbf{g})^\top\big) \quad = \quad \pi_2 T\big(\Gamma(p)\big)^{-1} \tilde{G}\Big(\gamma\big(p, u_0, (\mathbf{f}, \mathbf{g})^\top\big)\Big)$$

for all $\big(p, u_0, (\mathbf{f}, \mathbf{g})^\top\big)$ in the domain of F, with $\tilde{G}$ from Lemma 98, this statement follows from Lemma 98, Lemma 111, and the chain rule. We note that

$$T(\hat{P})(\mathbf{f}, \mathbf{g}, \mathbf{0})^\top \;=\; (\mathbf{f}, \mathbf{g}, \mathbf{0})^\top \qquad \text{and} \qquad \big[DT(\hat{P})P\big](\mathbf{f}, \mathbf{g}, \mathbf{0})^\top \;=\; 0\,,$$

$(\mathbf{f}, \mathbf{g})^\top \in W^{1,1}\big((0,t_1), L^2(D, \mathbb{R}^3 \times \mathbb{R}^{3\times 3}_{\mathrm{sym}})\big)$, $\hat{P} \in \Gamma(\mathcal{P})$, $P \in \mathcal{T}\,\Gamma(\mathcal{P})$. $\square$

Chapter 7

Adjoint Operators

The goal of this chapter is the derivation of the adjoint operator of the bounded linear operator, which is given by the Fréchet-derivative of the parameter-to-solution-map F from the previous chapter at a fixed point of its domain.

7.1 The Abstract Case

7.1.1 Auxiliaries

Throughout this section, let $\big(X, (\,.\,,.)\big)$ denote a real Hilbert space, $\|.\|$ the norm induced by $(\,.\,,.)$, and let $t_1 > 0$.

Lemma 114. *Let $R : X \supseteq D(R) \to X$ be a maximal monotone operator. Then the adjoint operator $R^* : X \supseteq D(R^*) \to X$ is maximal monotone, too.*

Proof. We adopt the argumentation in the proof of Lemma 3.1 in [14].

By Lemma 2(c) in section 3.1, the operator $\mathrm{Id}+\alpha R : \big(\mathcal{D}(R), \|.\|\big) \to \big(X, \|.\|\big)$ is boundedly invertible for every $\alpha > 0$. Consequently, also $\alpha\mathrm{Id}+R = \alpha\big(\mathrm{Id}+(1/\alpha)R\big)$ is boundedly invertible between these spaces for $\alpha > 0$. So the adjoint operator $\big((\alpha\mathrm{Id} + R)^{-1}\big)^* \in \mathcal{L}(X)$ exists. In the sequel we fix one arbitrary $\alpha > 0$.

It is

$$\begin{aligned}
\mathcal{D}(R^*) &= \big\{v \in X :\ u \mapsto (Ru, v) \text{ is continuous on } \mathcal{D}(R)\big\} \\
&= \big\{v \in X :\ u \mapsto (Ru, v) + (\alpha u, v) \text{ is continuous on } \mathcal{D}(R)\big\} \\
&= \mathcal{D}\big((\alpha\mathrm{Id} + R)^*\big)\,,
\end{aligned}$$

and

$$\begin{array}{rcl}
\mathcal{D}(R) & \to & \mathbb{R} \\
u & \to & \Big((\alpha\mathrm{Id} + R)u\,,\ \big((\alpha\mathrm{Id} + R)^{-1}\big)^* v\Big) \quad = \quad (u, v)
\end{array}$$

is continuous for every $v \in X$. So $\big((\alpha \mathrm{Id} + R)^{-1}\big)^* v \in \mathcal{D}(R^*)$ for every $v \in X$ and furthermore

$$\Big(u\,,\,(\alpha \mathrm{Id} + R^*)\big((\alpha \mathrm{Id} + R)^{-1}\big)^* v\Big) \quad = \quad (u, v)\,,$$

$v \in X$, $u \in \mathcal{D}(R)$. Since $\mathcal{D}(R)$ is dense in X by Lemma 2(a), it follows

$$(\alpha \mathrm{Id} + R^*)\big((\alpha \mathrm{Id} + R)^{-1}\big)^* \quad = \quad \mathrm{Id} \qquad \text{on } X\,. \tag{7.1}$$

Next we prove that $\alpha \mathrm{Id} + R^* : \mathcal{D}(R^*) \to X$ is injective. Therefore we assume that $(\alpha \mathrm{Id} + R^*)v = 0$ for some $v \in \mathcal{D}(R^*)$. Then

$$0 \quad = \quad \big((\alpha \mathrm{Id} + R^*)v\,,\,u\big) \quad = \quad \big(v\,,\,(\alpha \mathrm{Id} + R)u\big)$$

for every $u \in \mathcal{D}(R)$. As $\alpha \mathrm{Id} + R$ is onto, it follows $v = 0$. So $\alpha \mathrm{Id} + R^*$ is injective.

From (7.1) it follows

$$\begin{aligned}(\alpha \mathrm{Id} + R^*)\Big(\big((\alpha \mathrm{Id} + R)^{-1}\big)^*(\alpha \mathrm{Id} + R^*)v - v\Big) &= (\alpha \mathrm{Id} + R^*)v - (\alpha \mathrm{Id} + R^*)v \\ &= 0\end{aligned}$$

for all $v \in \mathcal{D}(R^*)$. So the injectivity of $\alpha \mathrm{Id} + R^*$ implies

$$\big((\alpha \mathrm{Id} + R)^{-1}\big)^*(\alpha \mathrm{Id} + R^*) \quad = \quad \mathrm{Id} \qquad \text{on } \mathcal{D}(R^*)\,.$$

Together with (7.1), we have that $\alpha \mathrm{Id} + R^* : \mathcal{D}(R^*) \to X$ is bijective with

$$(\alpha \mathrm{Id} + R^*)^{-1} \quad = \quad \big((\alpha \mathrm{Id} + R)^{-1}\big)^*\,.$$

In particular this holds for $\alpha = 1$.

So it remains to prove the monotonicity of R^*. For every $v \in \mathcal{D}(R^*)$ it is

$$\begin{aligned}&\Big((\alpha \mathrm{Id} + R^*)v\,,\,v\Big) \\ &\quad = \quad \Big((\alpha \mathrm{Id} + R^*)v\,,\,(\alpha \mathrm{Id} + R^*)^{-1}(\alpha \mathrm{Id} + R^*)v\Big) \\ &\quad = \quad \Big((\alpha \mathrm{Id} + R^*)v\,,\,\big((\alpha \mathrm{Id} + R)^{-1}\big)^*(\alpha \mathrm{Id} + R^*)v\Big) \\ &\quad = \quad \Big((\alpha \mathrm{Id} + R)^{-1}(\alpha \mathrm{Id} + R^*)v\,,\,(\alpha \mathrm{Id} + R^*)v\Big) \\ &\quad = \quad \Big(\big((\alpha \mathrm{Id} + R)^{-1}(\alpha \mathrm{Id} + R^*)v\big)\,,\,(\alpha \mathrm{Id} + R)\big((\alpha \mathrm{Id} + R)^{-1}(\alpha \mathrm{Id} + R^*)v\big)\Big) \\ &\quad \geq \quad 0\,.\end{aligned}$$

Since $\alpha > 0$ was arbitrary, we conclude $(R^*v, v) \geq 0$, $\;v \in \mathcal{D}(R^*)$. $\square$

The following lemma constitutes the core part of the proofs of Theorem 117 and Theorem 119 on adjoint operators to the derivatives of $\tilde{G}$ and H from section 6.1.2. The idea of its proof is taken from [14] again.

Lemma 115. *Let $R : X \supseteq \mathcal{D}(R) \to X$ be a maximal monotone operator, $v_0 \in X$, $g \in L^1\big((0,t_1),X\big)$, and*

$$v(t) \quad := \quad S_R(t)v_0 \; + \int_0^t S_R(t-s)g(s)\,ds\,, \qquad\qquad t \in [0,t_1]\,, \tag{7.2}$$

which is the mild solution of

$$\begin{aligned} v'(t) \quad &= \quad -Rv(t) \;+\; g(t)\,, \qquad\qquad t \in [0,t_1]\,,\\ v(0) \quad &= \quad v_0\,. \end{aligned}$$

Then for any $\varphi \in L^1\big((0,t_1),X\big)$, it holds

$$\int_0^{t_1} \big(v(t),\varphi(t)\big)\,dt \quad = \quad \int_0^{t_1} \big(g(t),\bar{v}(t)\big)\,dt \;+\; \big(v_0\,,\,\bar{v}(0)\big)\,,$$

where

$$\bar{v}(t) \quad := \quad \int_0^{t_1-t} S_{R^*}(t_1-t-s)\varphi(t_1-s)\,ds\,, \qquad\qquad t \in [0,t_1]\,. \tag{7.3}$$

So

$$\bar{v}(t_1-t) \quad = \quad \int_0^t S_{R^*}(t-s)\varphi(t_1-s)\,ds\,, \qquad\qquad t \in [0,t_1]\,,$$

which means, that $\bar{v}(t_1-\cdot)$ satisfies

$$\begin{aligned} \frac{d}{dt}\bar{v}(t_1-t) \quad &= \quad -R^*\bar{v}(t_1-t) \;+\; \varphi(t_1-t)\,, \qquad\qquad t \in [0,t_1]\,,\\ \bar{v}(t_1) \quad &= \quad 0 \end{aligned}$$

in the mild sense.

Proof. First, we note that (7.3) is well-defined, since by Lemma 114, the operator R^* is maximal monotone.

Let $\varphi \in L^1\big((0,t_1),X\big)$ be arbitrary. To be able to interpret both occurring evolution equations in the classical sense, we approximate g by a sequence $(g_n)_{n\in\mathbb{N}}$ and φ by a sequence $(\varphi_n)_{n\in\mathbb{N}}$, such that $g_n, \varphi_n \in C_c^\infty\big((0,t_1),X\big)$, $n \in \mathbb{N}$, and $\|g_n - g\|_{L^1((0,t_1),X)}, \|\varphi_n - \varphi\|_{L^1((0,t_1),X)} \to 0$, $n \to \infty$, which is possible according to Lemma A.3. Furthermore, due to Lemma 2(a), there is $(v_{0,n})_{n\in\mathbb{N}}$ in $\mathcal{D}(R)$ with $\|v_{0,n} - v_0\| \to 0$, $n \to \infty$.

Then for the functions v_n satisfying

$$\begin{aligned} v_n'(t) &= -Rv_n(t) + g_n(t)\,, \qquad t \in [0, t_1]\,, \\ v_n(0) &= v_{0,n} \end{aligned} \tag{7.4}$$

in the classical sense, it is

$$\big\|v_n(t) - v(t)\big\| \quad \leq \quad \|v_{0,n} - v_0\| + \int_0^t \big\|g_n(s) - g(s)\big\|\, ds\,,$$

$t \in [0, t_1]$, $n \in \mathbb{N}$. So $\|v_n - v\|_{C([0,t_1],X)} \to 0,\ n \to \infty$.

And analogously, for $\bar{v}_n$ satisfying

$$\begin{aligned} \frac{d}{dt}\bar{v}_n(t_1 - t) &= -R^*\bar{v}_n(t_1 - t) + \varphi_n(t_1 - t)\,, \qquad t \in [0, t_1]\,, \\ \bar{v}_n(t_1) &= 0 \end{aligned} \tag{7.5}$$

classically, it holds

$$\big\|\bar{v}_n(t_1 - t) - v(t_1 - t)\big\| \quad \leq \quad \int_0^t \big\|\varphi_n(t_1 - s) - \varphi(t_1 - s)\big\|\, ds\,,$$

$t \in [0, t_1]$, $n \in \mathbb{N}$. So $\|\bar{v}_n - \bar{v}\|_{C([0,t_1],X)} \to 0,\ n \to \infty$.

We now use $\varphi_n = -\bar{v}_n' + R^*\bar{v}_n$ and $\bar{v}_n(t_1) = 0$, which are equivalent to (7.5), and also (7.4) to calculate

$$\begin{aligned} &\int_0^{t_1} \big(v_n(t), \varphi_n(t)\big)\, dt \\ &\quad = \int_0^{t_1} \big(v_n(t),\ -\bar{v}_n'(t) + R^*\bar{v}_n(t)\big)\, dt \\ &\quad = \int_0^{t_1} \big(v_n'(t) + Rv_n(t),\ \bar{v}_n(t)\big)\, dt + \big(v_{0,n}, \bar{v}_n(0)\big) \\ &\quad = \int_0^{t_1} \big(g_n(t),\ \bar{v}_n(t)\big)\, dt + \big(v_{0,n}, \bar{v}_n(0)\big)\,, \end{aligned}$$

$n \in \mathbb{N}$. With $n \to \infty$, it follows

$$\int_0^{t_1} \big(v(t), \varphi(t)\big)\, dt \quad = \quad \int_0^{t_1} \big(g(t),\ \bar{v}(t)\big)\, dt + \big(v_0, \bar{v}(0)\big)\,.$$

□

7.1.2 Adjoints of the Derivatives

This section is to be understood in the context of section 6.1.2. In particular we suppose that Assumption 93 holds.

First we recall the map $\tilde{G}$ introduced in Lemma 98:

$$\begin{aligned} \tilde{G} : \big(U, \|.\|_{\mathcal{T}U,X}\big) \times \big(\mathcal{D}(\tilde{Q}), \|.\|_{\tilde{Q},X}\big) \times W^{1,1}\big((0,t_1),X\big) \quad &\to \quad C\big([0,t_1],X\big) \\ (P, u_0, f) \quad &\mapsto \quad G\big(P,\, T(P)u_0,\, T(P)f\big) \end{aligned}$$

with

$$\begin{aligned} G : \quad & \big(U, \|.\|_{\mathcal{T}U,X}\big) \;\times\; \big(\mathcal{D}(Q), \|.\|_{Q,X}\big) \;\times\; W^{1,1}\big((0,t_1),(X,\|.\|_X)\big) \\ & \qquad \to \quad C^1\Big([0,t_1],(X,\|.\|_X)\Big) \cap C\Big([0,t_1],\big(\mathcal{D}(Q),\|.\|_{Q,X}\big)\Big)\,, \\ & (P, w_0, f) \quad \mapsto \quad w\,, \end{aligned}$$

defined in (6.4), where w classically solves

$$w'(t) \;+\; \beta(P)\,w(t) \quad = \quad f(t)\,, \qquad t \in [0,t_1]\,, \qquad\qquad w(0) \;=\; w_0\,.$$

According to Lemma 98, $\tilde{G}$ is Fréchet-differentiable, and its derivative is of the form

$$\begin{aligned} & \big[D\tilde{G}(\hat{P},\hat{u}_0,\hat{f})(P,u_0,f)\big](t) \\ & \quad = \quad S_{\beta(\hat{P})}(t)\,\Big(\big[DT(\hat{P})P\big]\hat{u}_0 + T(\hat{P})u_0\Big) \\ & \qquad\quad + \int_0^t S_{\beta(\hat{P})}(t-s)\,\Big(\big[DT(\hat{P})P\big]\hat{f}(s) + T(\hat{P})f(s) \\ & \qquad\qquad\qquad\qquad\qquad - \beta(P)\tilde{G}(\hat{P},\hat{u}_0,\hat{f})(s)\Big)\,ds\,. \end{aligned} \tag{7.6}$$

Assumption 116. In the following, we interpret $\tilde{G}$ with the bigger codomain $L^2\big((0,t_1),X\big)$. Since $C\big([0,t_1],X\big)$ continuously embeds into $L^2\big((0,t_1),X\big)$, the differentiability of $\tilde{G}$ is preserved under this modification.

Moreover, for fixed $\hat{\Pi} = (\hat{P},\hat{u}_0,\hat{f}) \in U \times \mathcal{D}(\tilde{Q}) \times W^{1,1}\big((0,t_1),X\big)$, we endow $L^2\big((0,t_1),X\big)$ with the scalar product $\int_0^{t_1} \big(v(t)\,,w(t)\big)_{E,\hat{P}}\,dt$, $v,w \in L^2\big((0,t_1),X\big)$, which is equivalent to the canonical scalar product $\int_0^{t_1} \big(v(t)\,,w(t)\big)_{E,\hat{P}}\,dt$, $v,w \in L^2\big((0,t_1),X\big)$. Here, $(.\,,.)_{E,\hat{P}} = (\hat{P}_1^{-1}\,.\,,.)_X$ is the scalar product on X, with respect to which $\beta(\hat{P})$ is maximal monotone (see Assumption 82).

The Fréchet-derivative of $\tilde{G}$ at point $\hat{\Pi}$ is a bounded linear operator

$$\begin{aligned} D\tilde{G}(\hat{\Pi}) : \quad & \big(\mathcal{T}U, \|.\|_{\mathcal{T}U,X}\big) \times \big(\mathcal{D}(\tilde{Q}), \|.\|_{\tilde{Q},X}\big) \times W^{1,1}\Big((0,t_1),\big(X,(.\,,.)_X\big)\Big) \\ & \qquad \to \quad L^2\Big((0,t_1),\big(X,(.\,,.)_{E,\hat{P}}\big)\Big)\,. \end{aligned} \tag{7.7}$$

Yet, we make use of formula (7.6) to extend it to a bounded linear operator

$$\begin{aligned} D\tilde{G}(\hat{\Pi}): \quad & \big(\mathcal{T}U, \|.\|_{\mathcal{T}U,X}\big) \times \big(X, (.\,,.)_X\big) \times L^1\Big((0,t_1), \big(X, (.\,,.)_X\big)\Big) \\ \rightarrow \quad & L^2\Big((0,t_1), \big(X, (.\,,.)_{E,\hat{P}}\big)\Big) \end{aligned} \tag{7.8}$$

with respect to a weaker norm in the domain.

Its adjoint then is an operator

$$\begin{aligned} D\tilde{G}(\hat{\Pi})^*: \quad & L^2\Big((0,t_1), \big(X, (.\,,.)_{E,\hat{P}}\big)\Big) \\ & \rightarrow \quad \big(\mathcal{T}U, \|.\|_{\mathcal{T}U,X}\big)' \times \big(X, (.\,,.)_X\big) \times L^\infty\Big((0,t_1), \big(X, (.\,,.)_X\big)\Big), \end{aligned}$$

where we used the notation $\big(\mathcal{T}U, \|.\|_{\mathcal{T}U,X}\big)'$ for the dual space of the normed space $\big(\mathcal{T}U, \|.\|_{\mathcal{T}U,X}\big)$. □

To clarify the connection between the adjoint of the extended operator (7.8) and the adjoint of the original one (7.7), we argue as follows.

If we denote the operator (7.7) by R_1 and the operator (7.8) by R_2, then for any fixed $w \in L^2((0,t_1), X)$, the value $R_2^* w$ is a triple $(P', u_0', f') \in \mathcal{T}U' \times X \times L^\infty\big((0,t_1), X\big)$ with the property that for every $\Pi = (P, u_0, f) \in \mathcal{T}U \times X \times L^1\big((0,t_1), X\big)$, it is

$$\int_0^{t_1} (w\,, R_2\Pi)_{E,\hat{P}}\, dt \quad = \quad P'(P) \,+\, (u_0'\,, u_0)_X \,+\, \int_0^{t_1} \big(f'(t)\,, f(t)\big)_X\, dt\,.$$

On the other hand, the value $R_1^* w$ is a triple (P'', u_0'', f'') in the dual space of $\mathcal{T}U \times \mathcal{D}(\tilde{Q}) \times W^{1,1}\big((0,t_1), X\big)$, such that for every $\Pi = (P, u_0, f) \in \mathcal{T}U \times \mathcal{D}(\tilde{Q}) \times W^{1,1}\big((0,t_1), X\big)$ it is

$$\int_0^{t_1} (w\,, R_1\Pi)_{E,\hat{P}}\, dt \quad = \quad P''(P) \,+\, u_0''(u_0) \,+\, f''(f)\,.$$

Since $R_2\Pi = R_1\Pi$ for all $\Pi \in \mathcal{T}U \times \mathcal{D}(\tilde{Q}) \times W^{1,1}\big((0,t_1), X\big)$, it follows $P'' = P'$, $u_0'' = (u_0'\,, \cdot)_X\big|_{\mathcal{D}(\tilde{Q})}$ and $f'' = \int_0^{t_1} \big(f'(t)\,, \cdot\,(t)\big)_X\, dt\big|_{W^{1,1}((0,t_1),X)}$.

We turn to our first main result of this subsection.

Theorem 117. *The adjoint operator*

$$\begin{aligned} D\tilde{G}(\hat{P}, \hat{u}_0, \hat{f})^*: \quad & L^2\Big((0,t_1), \big(X, (.\,,.)_{E,\hat{P}}\big)\Big) \\ & \rightarrow \big(\mathcal{T}U, \|.\|_{\mathcal{T}U,X}\big)' \times \big(X, (.\,,.)_X\big) \times L^\infty\Big((0,t_1), \big(X, (.\,,.)_X\big)\Big) \end{aligned}$$

of the Fréchet-derivative

$$D\tilde{G}(\hat{P},\hat{u}_0,\hat{f}): \quad \big(\mathcal{T}U, \|.\|_{\mathcal{T}U,X}\big) \times \big(X, (\,.\,,.)_X\big) \times L^1\Big((0,t_1), \big(X,(\,.\,,.)_X\big)\Big) \\ \to\ L^2\Big((0,t_1), \big(X,(\,.\,,.)_{E,\hat{P}}\big)\Big)$$

of $\tilde{G}$ at some point $(\hat{P},\hat{u}_0,\hat{f}) \in U \times \mathcal{D}(\tilde{Q}) \times W^{1,1}\big((0,t_1),X\big)$ with respect to the scalar product $\int_0^{t_1}\big(v(t),w(t)\big)_{E,\hat{P}}\,dt$, $v,w \in L^2\big((0,t_1),X\big)$, has the representation

$$\begin{aligned}&\big[D\tilde{G}(\hat{P},\hat{u}_0,\hat{f})^*\varphi\big](P,u_0,f)\\ &= \int_0^{t_1}\Big(-\beta(P)\hat{w}(t) + \big[DT(\hat{P})P\big]\hat{f}(t) + T(\hat{P})f(t)\ ,\ \bar{w}(t)\Big)_{E,\hat{P}}\,dt \\ &\quad + \Big(\big[DT(\hat{P})P\big]\hat{u}_0 + T(\hat{P})u_0\ ,\ \bar{w}(0)\Big)_{E,\hat{P}}\end{aligned} \tag{7.9}$$

for every $\varphi \in L^2\big((0,t_1),X\big)$ and every $(P,u_0,f) \in \mathcal{T}U \times X \times L^1\big((0,t_1),X\big)$, where $\bar{w}$ satisfies

$$\begin{aligned}\frac{d}{dt}\bar{w}(t_1-t) &= -\beta(\hat{P})^*\bar{w}(t_1-t) + \varphi(t_1-t)\,, \qquad t\in[0,t_1]\,,\\ \bar{w}(t_1) &= 0\end{aligned}$$

in the mild sense with the adjoint $\beta(\hat{P})^*$ of $\beta(\hat{P})$ with respect to $(\,.\,,.)_{E,\hat{P}}$, and $\hat{w}$ satisfies

$$\begin{aligned}\hat{w}'(t) &= -\beta(\hat{P})\hat{w}(t) + T(\hat{P})\hat{f}(t)\,, \qquad t\in[0,t_1]\,,\\ \hat{w}(0) &= T(\hat{P})\hat{u}_0\end{aligned}$$

in the classical sense.

Proof. According to Lemma 115, with the role of v in (7.2) taken by the function $D\tilde{G}(\hat{P},\hat{u}_0,\hat{f})(P,u_0,f)$ in the form (7.6), the right-hand side of (7.9) is equal to

$$\int_0^{t_1}\Big(\big[D\tilde{G}(\hat{P},\hat{u}_0,\hat{f})(P,u_0,f)\big](t)\,,\,\varphi(t)\Big)_{E,\hat{P}}\,dt\,.$$

□

Next, we turn to the map

$$\begin{aligned}H:\ U\times\mathcal{D}(\tilde{Q})\times W^{1,1}\big((0,t_1),X\big) &\to C\big([0,t_1],X\big)\,,\\ H(P,u_0,f)(t) &= T(P)^{-1}\,G\Big(P,\,T(P)u_0,\,T(P)f\Big)(t)\end{aligned}$$

from Definition 95 with the codomain $C\big([0,t_1],X\big)$ as considered in Theorem 99. According to this theorem, it is differentiable, and its derivative is given by

$$\begin{aligned} DH(\hat\Pi): \quad & \big(\mathcal{T}U, \|.\|_{\mathcal{T}U,X}\big) \times \big(\mathcal{D}(\tilde Q), \|.\|_{\tilde Q,X}\big) \times W^{1,1}\Big((0,t_1), \big(X, (\,.\,,.)_X\big)\Big) \\ & \quad \to \quad L^2\Big((0,t_1), \big(X, (\,.\,,.)_{T(\hat P)}\big)\Big)\,, \\ & \big[DH(\hat P, \hat u_0, \hat f)(P, u_0, f)\big](t) \\ & \quad = \quad S_{\hat A}(t)u_0 \;+\; \int_0^t S_{\hat A}(t-s)\big(-A\hat u(s) + f(s)\big)\,ds\,, \end{aligned} \tag{7.10}$$

$t \in [0,t_1]$, where

$$\hat A \quad = \quad T^{-1}\beta(\hat P)T \tag{7.11}$$

with $T := T(\hat P)$, $T^{-1} := T(\hat P)^{-1}$, $T_1 := DT(\hat P)^{-1}P$,

$$A \quad = \quad T_1\beta(\hat P)T \;-\; \hat A T_1 T \;+\; T^{-1}\beta(P)T\,, \tag{7.12}$$

and $\hat u$ is the classical solution of

$$\begin{aligned} \hat u'(t) &\quad=\quad -\hat A\hat u(t) \;+\; \hat f(t)\,, \qquad\qquad t \in [0,t_1]\,, \\ \hat u(0) &\quad=\quad \hat u_0\,. \end{aligned}$$

Assumption 118. Let $\hat\Pi = (\hat P, \hat u_0, \hat f) \in U \times \mathcal{D}(\tilde Q) \times W^{1,1}\big((0,t_1),X\big)$ be arbitrary and fixed. As done in the case of $\tilde G$, we interpret H and $DH(\hat\Pi)$ as maps with codomain $L^2\big((0,t_1),X\big)$. This time, however, we endow $L^2\big((0,t_1),X\big)$ with the scalar product $\int_0^{t_1}\big(u(t)\,,v(t)\big)_{T(\hat P)}\,dt$, $u,v \in L^2\big((0,t_1),X\big)$, where $(\,.\,,.)_{T(\hat P)} = \big(T(\hat P)\,.\,,T(\hat P)\,.\big)_{E,\hat P}$ is the energy scalar product on X, with respect to which the operator $\hat A$ in (7.11) is maximal monotone (see Theorem 28).

In further analogy to the case of $\tilde G$, we use formula (7.10) to extend $DH(\hat\Pi)$ to the bounded linear operator

$$\begin{aligned} DH(\hat\Pi): \quad & \big(\mathcal{T}U, \|.\|_{\mathcal{T}U,X}\big) \times \big(X, (\,.\,,.)_X\big) \times L^1\Big((0,t_1), \big(X, (\,.\,,.)_X\big)\Big) \\ & \quad \to \quad L^2\Big((0,t_1), \big(X, (\,.\,,.)_{T(\hat P)}\big)\Big) \end{aligned}$$

with respect to a weaker norm in the domain. □

Theorem 119. *The adjoint operator*

$$\begin{aligned} DH(\hat P, \hat u_0, \hat f)^*: \quad & L^2\Big((0,t_1), \big(X, (\,.\,,.)_{T(\hat P)}\big)\Big) \\ & \to \big(\mathcal{T}U, \|.\|_{\mathcal{T}U,X}\big)' \times \big(X, (\,.\,,.)_X\big) \times L^\infty\Big((0,t_1), \big(X, (\,.\,,.)_X\big)\Big) \end{aligned}$$

of the Fréchet-derivative

$$DH(\hat{P},\hat{u}_0,\hat{f}) : \quad \big(\mathcal{T}U, \|.\|_{\mathcal{T}U,X}\big) \times \big(X,(\,.\,,.)_X\big) \times L^1\Big((0,t_1),\big(X,(\,.\,,.)_X\big)\Big) \\ \to L^2\Big((0,t_1),\big(X,(\,.\,,.)_{T(\hat{P})}\big)\Big)$$

of H at some point $(\hat{P},\hat{u}_0,\hat{f}) \in U \times \mathcal{D}(\tilde{Q}) \times W^{1,1}\big((0,t_1),X\big)$ with respect to the scalar product $\int_0^{t_1} \big(u(t),v(t)\big)_{T(\hat{P})}\,dt$, $u,v \in L^2\big((0,t_1),X\big)$, has the representation

$$\begin{aligned} &\big[DH(\hat{P},\hat{u}_0,\hat{f})^*\varphi\big](P,u_0,f) \\ &\quad = \int_0^{t_1} \big(-A\hat{u}(t)+f(t)\,,\ \bar{u}(t)\big)_{T(\hat{P})}\,dt \ + \ \big(u_0,\,\bar{u}(0)\big)_{T(\hat{P})} \end{aligned} \tag{7.13}$$

for every $\varphi \in L^2\big((0,t_1),X\big)$ and every $(P,u_0,f) \in \mathcal{T}U \times X \times L^1\big((0,t_1),X\big)$, with A from (7.12) *and where $\bar{u}$ satisfies*

$$\begin{aligned} \frac{d}{dt}\bar{u}(t_1-t) &= -\hat{A}^*\bar{u}(t_1-t) \ + \ \varphi(t_1-t)\,, \qquad t\in[0,t_1]\,, \\ \bar{u}(t_1) &= 0 \end{aligned}$$

in the mild sense, where $\hat{A}^*$ is the adjoint of $\hat{A}$ from (7.11) *with respect to $(\,.\,,.)_{T(\hat{P})}$, and $\hat{u}$ satisfies*

$$\begin{aligned} \hat{u}'(t) &= -\hat{A}\hat{u}(t) \ + \ \hat{f}(t)\,, \qquad t\in[0,t_1]\,, \\ \hat{u}(0) &= \hat{u}_0 \end{aligned}$$

in the classical sense.

Proof. Analogously to the proof of the previous theorem, it follows from Lemma 115, where the role of v in (7.2) is taken by $DH(\hat{P},\hat{u}_0,\hat{f})(P,u_0,f)$ in the form (7.10) this time, that the expression on the right-hand side of (7.13) is equal to

$$\int_0^{t_1} \Big(\big[DH(\hat{P},\hat{u}_0,\hat{f})(P,u_0,f)\big](t)\,,\ \varphi(t)\Big)_{T(\hat{P})}\,dt\,.$$

□

7.2 Viscoelasticity

7.2.1 Adjoint Generators

In this section we derive the adjoint operators to B as in (5.29) and A introduced in (2.14) with respect to their corresponding energy scalar products. By X we still denote the function space defined in (5.4),

$$X \quad = \quad L^2(D,\mathbb{R}^3) \times L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}}) \times L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}})^L\,.$$

First, we recall $B : X \supseteq \mathcal{D}(B) \to X$ with

$$Bw \quad = \quad \begin{pmatrix} -\vartheta \operatorname{div}\left(\boldsymbol{\sigma}_H + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l}\right) \\ -C\big(\mu_H, \kappa_H\big)\,\varepsilon(\mathbf{v}) \\ -C\big(\mu_{M,1}, \kappa_{M,1}\big)\,\varepsilon(\mathbf{v}) + \omega_{\boldsymbol{\sigma},1}\boldsymbol{\sigma}_{M,1} \\ \vdots \\ -C\big(\mu_{M,L}, \kappa_{M,L}\big)\,\varepsilon(\mathbf{v}) + \omega_{\boldsymbol{\sigma},L}\boldsymbol{\sigma}_{M,L} \end{pmatrix}, \tag{7.14}$$

$w = (\mathbf{v}, \boldsymbol{\sigma}_H, \boldsymbol{\sigma}_M)^\top \in \mathcal{D}(B)$, and

$$\mathcal{D}(B) \quad = \quad \Big\{(\mathbf{v}, \boldsymbol{\sigma}_H, \boldsymbol{\sigma}_M)^\top \in\ V \times L^2(D, \mathbb{R}^{3\times 3}_{\mathrm{sym}}) \times L^2(D, \mathbb{R}^{3\times 3}_{\mathrm{sym}})^L\ : \\ \boldsymbol{\sigma}_H + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l} \in S\Big\} \tag{7.15}$$

as stated in (5.28), where V was defined in (5.9) as

$$V \quad = \quad \overline{\Big\{\ \boldsymbol{\varphi} \in C^\infty(D, \mathbb{R}^3) \cap H(\varepsilon, D, \mathbb{R}^3)\ :\quad \partial D_D \subseteq \mathbb{R}^3 \setminus \operatorname{supp}(\boldsymbol{\varphi})\ \Big\}}^{\|.\|_V} \tag{7.16}$$

with $\|.\|_V$ denoting the canonical norm in $H(\varepsilon, D, \mathbb{R}^3)$, and S was defined in (5.10) as

$$S \ = \ \Big\{\ \boldsymbol{\sigma} \in H(\operatorname{div}, D, \mathbb{R}^{3\times 3}_{\mathrm{sym}}) :\ \forall \boldsymbol{\varphi} \in V :\ \int_D \varepsilon(\boldsymbol{\varphi}) : \boldsymbol{\sigma}\ +\ \boldsymbol{\varphi} \cdot \operatorname{div} \boldsymbol{\sigma}\, dx = 0\ \Big\}. \tag{7.17}$$

The corresponding energy scalar product was defined in (5.32) and is of the form

$$(w_1, w_2)_E \quad = \quad \Big(\frac{1}{\vartheta}\mathbf{v}^{(1)}\,,\,\mathbf{v}^{(2)}\Big)_{L^2(D,\mathbb{R}^3)} \ + \ \Big(C\Big(\frac{1}{\mu_H}, \frac{1}{\kappa_H}\Big)\boldsymbol{\sigma}_H^{(1)}\,,\,\boldsymbol{\sigma}_H^{(2)}\Big)_{L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}})} \\ + \ \sum_{l=1}^{L} \Big(C\Big(\frac{1}{\mu_{M,l}}\,,\,\frac{1}{\kappa_{M,l}}\Big)\boldsymbol{\sigma}_{M,l}^{(1)}\,,\,\boldsymbol{\sigma}_{M,l}^{(2)}\Big)_{L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}})}, \tag{7.18}$$

$w_i = \big(\mathbf{v}^{(i)}, \boldsymbol{\sigma}_H^{(i)}, \boldsymbol{\sigma}_M^{(i)}\big)^\top \in X, \quad i = 1, 2.$

Theorem 120. *The adjoint of the operator $B : X \supseteq \mathcal{D}(B) \to X$ in (7.14) with $\mathcal{D}(B)$ as in (7.15), with respect to the scalar product $(.\,,.)_E$ in (7.18) is given by $B^* : X \supseteq \mathcal{D}(B^*) \to X$, where*

$$\mathcal{D}(B^*) \quad = \quad \Big\{(\mathbf{v}, \boldsymbol{\sigma}_H, \boldsymbol{\sigma}_M)^\top \in\ \tilde{V} \times L^2(D, \mathbb{R}^{3\times 3}_{\mathrm{sym}}) \times L^2(D, \mathbb{R}^{3\times 3}_{\mathrm{sym}})^L\ : \\ \boldsymbol{\sigma}_H + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l} \in S\Big\}$$

with

$$\tilde{V} \;:=\; \Big\{ \mathbf{v} \in H(\varepsilon, D, \mathbb{R}^{3\times 3}_{\mathrm{sym}}) :\; \forall \boldsymbol{\psi} \in S : \int_D \varepsilon(\mathbf{v}) : \boldsymbol{\psi} + \mathbf{v} \cdot \operatorname{div} \boldsymbol{\psi} \, dx = 0 \Big\} \tag{7.19}$$

and

$$B^* w \;=\; \begin{pmatrix} \vartheta \operatorname{div} \Big(\boldsymbol{\sigma}_H + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l} \Big) \\ C(\mu_H \,,\, \kappa_H)\, \varepsilon(\mathbf{v}) \\ C(\mu_{M,1} \,,\, \kappa_{M,1})\, \varepsilon(\mathbf{v}) \;+\; \omega_{\boldsymbol{\sigma},1}\, \boldsymbol{\sigma}_{M,1} \\ \vdots \\ C(\mu_{M,L} \,,\, \kappa_{M,L})\, \varepsilon(\mathbf{v}) \;+\; \omega_{\boldsymbol{\sigma},L}\, \boldsymbol{\sigma}_{M,L} \end{pmatrix} \tag{7.20}$$

for $w = (\mathbf{v}, \boldsymbol{\sigma}_H, \boldsymbol{\sigma}_M)^\top \in \mathcal{D}(B^*)$.

Proof. By the definition of the domain of an adjoint operator, the following statement holds true. For any $w_2 = (\mathbf{v}^{(2)}, \boldsymbol{\sigma}_H^{(2)}, \boldsymbol{\sigma}_M^{(2)})^\top \in X$ it is $w_2 \in \mathcal{D}(B^*)$, iff there is $z = (\mathbf{f}, \mathbf{g}, \mathbf{h})^\top (= B^* w_2) \in X$, such that for all $w_1 = (\mathbf{v}^{(1)}, \boldsymbol{\sigma}_H^{(1)}, \boldsymbol{\sigma}_M^{(1)})^\top \in \mathcal{D}(B)$ it holds

$$\begin{aligned}
&\int_D -\varepsilon(\mathbf{v}^{(1)}) : \Big(\boldsymbol{\sigma}_H^{(2)} + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l}^{(2)} \Big) - \operatorname{div} \Big(\boldsymbol{\sigma}_H^{(1)} + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l}^{(1)} \Big) \cdot \mathbf{v}^{(2)} \\
&\qquad\qquad + \sum_{l=1}^{L} \omega_{\boldsymbol{\sigma},l} C\Big(\frac{1}{\mu_{M,l}}, \frac{1}{\kappa_{M,l}} \Big) \boldsymbol{\sigma}_{M,l}^{(1)} : \boldsymbol{\sigma}_{M,l}^{(2)} \, dx \\
=\;& (B w_1, w_2)_E \\
=\;& (w_1, z)_E \\
=\;& \int_D \mathbf{v}^{(1)} \cdot \Big(\frac{1}{\vartheta} \mathbf{f} \Big) + \boldsymbol{\sigma}_H^{(1)} : C\Big(\frac{1}{\mu_H}, \frac{1}{\kappa_H} \Big) \mathbf{g} \\
&\qquad\qquad + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l}^{(1)} : C\Big(\frac{1}{\mu_{M,l}}, \frac{1}{\kappa_{M,l}} \Big) \mathbf{h}_l \, dx \,.
\end{aligned} \tag{7.21}$$

So in particular, $w_2 \in \mathcal{D}(B^*)$ implies, that this statement holds for $\boldsymbol{\sigma}_H^{(1)} = \mathbf{0}$, $\boldsymbol{\sigma}_{M,l}^{(1)} = \mathbf{0}$, $l = 1, \ldots, L$, and $\mathbf{v}^{(1)} \in C_c^\infty(D, \mathbb{R}^3)$. According to the definition of div in weak form in Notation 41, this means that $\boldsymbol{\sigma}_H^{(2)} + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l}^{(2)} \in H(\operatorname{div}, D, \mathbb{R}^{3\times 3}_{\mathrm{sym}})$, and $(1/\vartheta)\mathbf{f} = \operatorname{div} \big(\boldsymbol{\sigma}_H^{(2)} + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l}^{(2)} \big)$. Hence,

$$\int_D \varepsilon(\mathbf{v}^{(1)}) : \Big(\boldsymbol{\sigma}_H^{(2)} + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l}^{(2)} \Big) + \mathbf{v}^{(1)} \cdot \operatorname{div} \Big(\boldsymbol{\sigma}_H^{(2)} + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l}^{(2)} \Big) \, dx \;=\; 0$$

for all $\mathbf{v}^{(1)} \in V$, which by (7.17) is equivalent to $\boldsymbol{\sigma}_H^{(2)} + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l}^{(2)} \in S$.

Analogously, $w_2 \in \mathcal{D}(B^*)$ implies, that this statement holds true for the special case $\mathbf{v}^{(1)} = \mathbf{0}$, $\boldsymbol{\sigma}_{M,l}^{(1)} = \mathbf{0}$, $l = 1, \dots, L$, and $\boldsymbol{\sigma}_H^{(1)} \in C_c^\infty(D, \mathbb{R}^{3\times 3}_{\mathrm{sym}})$. From the definition of ε in weak form in Notation 41 it therefore follows, that $\mathbf{v}^{(2)} \in H(\varepsilon, D, \mathbb{R}^3)$, and $C(1/\mu_H, 1/\kappa_H)\mathbf{g} = \varepsilon(\mathbf{v}^{(2)})$. Hence,

$$\int_D \varepsilon(\mathbf{v}^{(2)}) : \boldsymbol{\sigma}_H^{(1)} + \mathbf{v}^{(2)} \cdot \operatorname{div} \boldsymbol{\sigma}_H^{(1)} \, dx \quad = \quad 0$$

for all $\boldsymbol{\sigma}_H^{(1)} \in S$, which is equivalent to $\mathbf{v}^{(2)} \in \tilde{V}$.

Finally, $w_2 \in \mathcal{D}(B^*)$ implies, that this statement holds true for the special case $\mathbf{v}^{(1)} = \mathbf{0}$, $\boldsymbol{\sigma}_H^{(1)} = \mathbf{0}$, $\boldsymbol{\sigma}_{M,l}^{(1)} = \mathbf{0}$, $l \in \{1, \dots, L\} \setminus \{k\}$ for one $k \in \{1, \dots, L\}$, and $\boldsymbol{\sigma}_{M,k}^{(1)} \in C_c^\infty(D, \mathbb{R}^{3\times 3}_{\mathrm{sym}})$. Again, the definition of ε in weak form implies, that $C(1/\mu_{M,k}, 1/\kappa_{M,k})(\mathbf{h}_k - \omega_{\boldsymbol{\sigma},k}\boldsymbol{\sigma}_{M,k}^{(2)}) = \varepsilon(\mathbf{v}^{(2)})$.

So

$$B^* w_2 \quad = \quad \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \\ \mathbf{h} \end{pmatrix} \quad = \quad \begin{pmatrix} \vartheta \operatorname{div} \left(\boldsymbol{\sigma}_H^{(2)} + \sum_{l=1}^L \boldsymbol{\sigma}_{M,l}^{(2)} \right) \\ C(\mu_H, \kappa_H)\varepsilon(\mathbf{v}^{(2)}) \\ C(\mu_{M,1}, \kappa_{M,1})\varepsilon(\mathbf{v}^{(2)}) + \omega_{\boldsymbol{\sigma},1}\boldsymbol{\sigma}_{M,1}^{(2)} \\ \vdots \\ C(\mu_{M,L}, \kappa_{M,L})\varepsilon(\mathbf{v}^{(2)}) + \omega_{\boldsymbol{\sigma},L}\boldsymbol{\sigma}_{M,L}^{(2)} \end{pmatrix}.$$

□

Next we turn to $A : X \supseteq \mathcal{D}(A) \to X$ with

$$Au \quad = \quad \begin{pmatrix} -\vartheta \operatorname{div} \boldsymbol{\sigma} \\ -C\Big(\mu_H + \sum_{l=1}^L \mu_{M,l} \,,\; \kappa_H + \sum_{l=1}^L \kappa_{M,l}\Big)\, \varepsilon(\mathbf{v}) \;-\; \sum_{l=1}^L \boldsymbol{\eta}_l \\ \omega_{\boldsymbol{\sigma},1} C\big(\mu_{M,1} \,,\; \kappa_{M,1}\big)\, \varepsilon(\mathbf{v}) \;+\; \omega_{\boldsymbol{\sigma},1}\boldsymbol{\eta}_1 \\ \vdots \\ \omega_{\boldsymbol{\sigma},L} C\big(\mu_{M,L} \,,\; \kappa_{M,L}\big)\, \varepsilon(\mathbf{v}) \;+\; \omega_{\boldsymbol{\sigma},L}\boldsymbol{\eta}_L \end{pmatrix}, \tag{7.22}$$

$u = (\mathbf{v}, \boldsymbol{\sigma}, \boldsymbol{\eta})^\top \in \mathcal{D}(A)$, and

$$\mathcal{D}(A) \quad = \quad V \times S \times L^2(D, \mathbb{R}^{3\times 3}_{\mathrm{sym}})^L \tag{7.23}$$

as defined in (5.11) where V is defined in (5.9) and S is defined in (5.10). The corresponding energy scalar product has been defined in (5.33) and is of the form

$$
\begin{aligned}
&(u_1, u_2)_T \\
&\quad = \quad \Big(\frac{1}{\vartheta}\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\Big)_{L^2(D,\mathbb{R}^3)} \\
&\qquad + \ \bigg(C\Big(\frac{1}{\mu_H}, \frac{1}{\kappa_H}\Big)\Big(\boldsymbol{\sigma}^{(1)} + \sum_{l=1}^{L}\frac{1}{\omega_{\boldsymbol{\sigma},l}}\boldsymbol{\eta}_l^{(1)}\Big)\ ,\ \boldsymbol{\sigma}^{(2)} + \sum_{l=1}^{L}\frac{1}{\omega_{\boldsymbol{\sigma},l}}\boldsymbol{\eta}_l^{(2)}\bigg)_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})} \\
&\qquad + \ \sum_{l=1}^{L}\bigg(\frac{1}{\omega_{\boldsymbol{\sigma},l}^2}C\Big(\frac{1}{\mu_{M,l}}\ ,\ \frac{1}{\kappa_{M,l}}\Big)\boldsymbol{\eta}_l^{(1)}\ ,\ \boldsymbol{\eta}_l^{(2)}\bigg)_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})}\ ,
\end{aligned}
\tag{7.24}
$$

$u_i = \big(\mathbf{v}^{(i)}, \boldsymbol{\sigma}^{(i)}, \boldsymbol{\eta}^{(i)}\big)^\top \in X, \quad i = 1, 2.$

Theorem 121. *The adjoint of the operator* $A : X \supseteq \mathcal{D}(A) \to X$ *in* (7.22) *with* $\mathcal{D}(A)$ *as in* (7.23)*, with respect to the scalar product* $(.\,,.)_T$ *in* (7.24) *is given by* $A^* : X \supseteq \mathcal{D}(A^*) \to X$*, where*

$$\mathcal{D}(A^*) \quad = \quad \tilde{V} \times S \times L^2(D, \mathbb{R}^{3\times3}_{\mathrm{sym}})^L$$

with $\tilde{V}$ *from* (7.19) *and* S *as in* (7.17)*, and*

$$
A^*u \quad = \quad \begin{pmatrix} \vartheta \operatorname{div} \boldsymbol{\sigma} \\ C\Big(\mu_H + \sum_{l=1}^{L}\mu_{M,l}\ ,\ \kappa_H + \sum_{l=1}^{L}\kappa_{M,l}\Big)\,\varepsilon(\mathbf{v})\ -\ \sum_{l=1}^{L}\boldsymbol{\eta}_l \\ -\,\omega_{\boldsymbol{\sigma},1}C\big(\mu_{M,1}\ ,\ \kappa_{M,1}\big)\,\varepsilon(\mathbf{v})\ +\ \omega_{\boldsymbol{\sigma},1}\boldsymbol{\eta}_1 \\ \vdots \\ -\,\omega_{\boldsymbol{\sigma},L}C\big(\mu_{M,L}\ ,\ \kappa_{M,L}\big)\,\varepsilon(\mathbf{v})\ +\ \omega_{\boldsymbol{\sigma},L}\boldsymbol{\eta}_L \end{pmatrix}, \tag{7.25}
$$

$u = (\mathbf{v}, \boldsymbol{\sigma}, \boldsymbol{\eta})^\top \in \mathcal{D}(A^*).$

Proof. Due to Lemma 32, it is $\mathcal{D}(A^*) = T^{-1}\mathcal{D}(B^*)$ and $A^* = T^{-1}B^*T$, where in this case

$$
T\begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma} \\ \boldsymbol{\eta}_1 \\ \vdots \\ \boldsymbol{\eta}_L \end{pmatrix} = \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma} + \sum_{l=1}^{L}\frac{1}{\omega_{\boldsymbol{\sigma},l}}\boldsymbol{\eta}_l \\ -\frac{1}{\omega_{\boldsymbol{\sigma},1}}\boldsymbol{\eta}_1 \\ \vdots \\ -\frac{1}{\omega_{\boldsymbol{\sigma},L}}\boldsymbol{\eta}_L \end{pmatrix}, \qquad T^{-1}\begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma}_H \\ \boldsymbol{\sigma}_{M,1} \\ \vdots \\ \boldsymbol{\sigma}_{M,L} \end{pmatrix} = \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma}_H + \sum_{l=1}^{L}\boldsymbol{\sigma}_{M,l} \\ -\omega_{\boldsymbol{\sigma},1}\boldsymbol{\sigma}_{M,1} \\ \vdots \\ -\omega_{\boldsymbol{\sigma},L}\boldsymbol{\sigma}_{M,L} \end{pmatrix},
$$

as introduced and calculated in (5.24) and (5.25), respectively. □

Remark 122. *It follows from the definition of S in (7.17), that for V in (7.16) and $\tilde{V}$ in (7.19), it is $V \subseteq \tilde{V}$. We leave the question open at this point, under which regularity assumptions on the boundary of D, it even holds $V = \tilde{V}$. Either way, with a look at (5.6) and Remark 47, we see that for $\mathbf{v} \in H(\varepsilon, D, \mathbb{R}^3)$, the condition $\mathbf{v} \in \tilde{V}$ can be interpreted as $\mathbf{v}\big|_{\partial D_D} = \mathbf{0}$ in a variational sense.*

If we compare the form of B in (7.14) and B^* in (7.20), we see that these operators only differ by some minus signs. Also A in (7.22) and A^* in (7.25) only differ by some minus signs. For numerical calculations this can be of advantage, since any numerical algorithm implemented to evaluate one of these operators can after only small adjustments be reused to evaluate the respective adjoint operator, too. It turns out that the key idea to accomplish this, consists in the choice of the natural energy scalar products $(.\,,.)_E$ and $(.\,,.)_T$. □

7.2.2 Adjoint of the Derivative

In this section we derive representations of the adjoint operator of the derivative $DF\big(\hat{p}, \hat{u}_0, (\hat{\mathbf{f}}, \hat{\mathbf{g}})^\top\big)$ of the parameter-to-solution-map F at any fixed point $\big(\hat{p}, \hat{u}_0, (\hat{\mathbf{f}}, \hat{\mathbf{g}})^\top\big)$ of its domain. We express it in terms of the original variables $\mathbf{v}$, $\boldsymbol{\sigma}$, $\boldsymbol{\eta}$ as well as the transformed ones $\mathbf{v}$, $\boldsymbol{\sigma}_H$, $\boldsymbol{\sigma}_M$. Concerning the notation and any preliminary assumptions, this section is to be understood in the context of section 6.2.

First we recall the structure of the parameter-to-solution-map. It is

$$\begin{aligned} F : \big(\mathcal{P}, \|.\|_{L^\infty(D,\mathbb{R}^{3+3L})}\big) \times \big(\mathcal{D}(\tilde{Q}), \|.\|_{\tilde{Q},X}\big) \times W^{1,1}\big((0,t_1),\, L^2(D, \mathbb{R}^3 \times \mathbb{R}^{3\times 3}_{\mathrm{sym}})\big) \\ \to \quad C\big([0,t_1]\,,\, L^2(D, \mathbb{R}^3 \times \mathbb{R}^{3\times 3}_{\mathrm{sym}})\big)\,, \\ F\big(p, u_0, (\mathbf{f}, \mathbf{g})^\top\big) \quad = \quad \pi_2\Big(H\Big(\gamma\big(p, u_0, (\mathbf{f}, \mathbf{g})^\top\big)\Big)\Big)\,, \end{aligned}$$

where here and in the following, we abbreviate $p = (\vartheta, \ldots, \omega_{\boldsymbol{\sigma},L})$. Furthermore, we are going to use the notation $\hat{p} = (\hat{\vartheta}, \ldots, \hat{\omega}_{\boldsymbol{\sigma},L})$, $\hat{P} = \Gamma(\hat{p})$ and $P = D\Gamma p$.

For any $\big(\hat{p}, \hat{u}_0, (\hat{\mathbf{f}}, \hat{\mathbf{g}})^\top\big) \in \mathcal{P} \times \mathcal{D}(\tilde{Q}) \times W^{1,1}\big((0,t_1), L^2(D, \mathbb{R}^3 \times \mathbb{R}^{3\times 3}_{\mathrm{sym}})\big)$, the derivative of the parameter-to-solution-map has the form

$$DF\big(\hat{p}, \hat{u}_0, (\hat{\mathbf{f}}, \hat{\mathbf{g}})^\top\big) \quad = \quad \pi_2 DH\Big(\gamma\big(\hat{p}, \hat{u}_0, (\hat{\mathbf{f}}, \hat{\mathbf{g}})^\top\big)\Big) D\gamma\big(\hat{p}, \hat{u}_0, (\hat{\mathbf{f}}, \hat{\mathbf{g}})^\top\big)\,.$$

Equivalently, we also wrote F as

$$F\big(p, u_0, (\mathbf{f}, \mathbf{g})^\top\big) \quad = \quad \pi_2 T\big(\Gamma(p)\big)^{-1} \tilde{G}\Big(\gamma\big(p, u_0, (\mathbf{f}, \mathbf{g})^\top\big)\Big)$$

for all $\big(p, u_0, (\mathbf{f}, \mathbf{g})^\top\big) \in \mathcal{P} \times \mathcal{D}(\tilde{Q}) \times W^{1,1}\big((0,t_1), L^2(D, \mathbb{R}^3 \times \mathbb{R}^{3\times 3}_{\mathrm{sym}})\big)$, where $p \mapsto \pi_2 T\big(\Gamma(p)\big)^{-1}$ is constant. So its derivative has the alternative form

$$DF\big(\hat{p}, \hat{u}_0, (\hat{\mathbf{f}}, \hat{\mathbf{g}})^\top\big) \quad = \quad \pi_2 T\big(\Gamma(\hat{p})\big)^{-1} D\tilde{G}\Big(\gamma\big(\hat{p}, \hat{u}_0, (\hat{\mathbf{f}}, \hat{\mathbf{g}})^\top\big)\Big) D\gamma\big(\hat{p}, \hat{u}_0, (\hat{\mathbf{f}}, \hat{\mathbf{g}})^\top\big)\,.$$

Using these two representations of the derivative, we get the two different representations of the adjoint derivative. The crucial parts of them are given by Theorem 117 on the adjoint of $D\tilde{G}(\hat{P})$ with respect to the energy scalar product $(\,.\,,.)_{E,\hat{P}}$ on X and Theorem 119 on the adjoint of $DH(\hat{P})$ with respect to the energy scalar product $(\,.\,,.)_{T(\hat{P})}$ on X, where $\hat{P} = \Gamma(\hat{p})$ is the pair of parameter operators at the material parameter coordinate $\hat{p} \in \mathcal{P}$, where we take the derivative of F. The remaining parts, that is the adjoints of π_2, $D\gamma\big(\hat{p}, \hat{u}_0, (\hat{\mathbf{f}}, \hat{\mathbf{g}})^\top\big)$ and $\pi_2 T\big(\Gamma(\hat{p})\big)^{-1}$, are provided by the following lemma. Before we formulate it, we make several adjustments to the domains and codomains of these maps in the subsequent assumption.

Assumption 123. In this section, we consider F to be a map with the bigger codomain $L^2\big(D \times (0,t_1),\ \mathbb{R}^3 \times \mathbb{R}^{3\times3}_{\mathrm{sym}}\big)$.

Furthermore, we extend

$$\begin{aligned} \pi_2 : \quad L^2\Big((0,t_1), \big(X, (\,.\,,.)_{T(\hat{P})}\big)\Big) \quad &\to \quad L^2\big(D \times (0,t_1),\ \mathbb{R}^3 \times \mathbb{R}^{3\times3}_{\mathrm{sym}}\big)\,, \\ (\mathbf{v}, \boldsymbol{\sigma}, \boldsymbol{\eta})^\top \quad &\mapsto \quad (\mathbf{v}, \boldsymbol{\sigma})^\top\,, \end{aligned} \tag{7.26}$$

$$\begin{aligned} \pi_2 T\big(\Gamma(\hat{p})\big)^{-1} :\ L^2\Big((0,t_1), \big(X, (\,.\,,.)_{E,\hat{P}}\big)\Big) \ &\to\ L^2\big(D \times (0,t_1),\ \mathbb{R}^3 \times \mathbb{R}^{3\times3}_{\mathrm{sym}}\big)\,, \\ (\mathbf{v}, \boldsymbol{\sigma}_H, \boldsymbol{\sigma}_M)^\top \ &\mapsto\ \Big(\mathbf{v},\ \boldsymbol{\sigma}_H + \sum_{l=1}^{L} \boldsymbol{\sigma}_{M,l}\Big)^\top \end{aligned} \tag{7.27}$$

and

$$\begin{aligned} D\gamma\big(\hat{p}, \hat{u}_0, (\hat{\mathbf{f}}, \hat{\mathbf{g}})^\top\big) : \quad & L^\infty(D, \mathbb{R}^{3+3L}) \times X \times L^1\big((0,t_1),\ L^2(D, \mathbb{R}^3 \times \mathbb{R}^{3\times3}_{\mathrm{sym}})\big) \\ &\quad \to \quad \mathcal{T}\,\Gamma(\mathcal{P}) \times X \times L^1\big((0,t_1), X\big)\,, \\ \big(p, u_0, (\mathbf{f}, \mathbf{g})^\top\big) \quad &\mapsto \quad \big(D\Gamma p, u_0, (\vartheta\hat{\mathbf{f}} + \hat{\vartheta}\mathbf{f}, \mathbf{g}, \mathbf{0})^\top\big)\,, \end{aligned} \tag{7.28}$$

and note that all three operators are bounded with respect to the new norms again.

Using these operators together with the extensions of $D\tilde{G}\Big(\gamma\big(\hat{p}, \hat{u}_0, (\hat{\mathbf{f}}, \hat{\mathbf{g}})^\top\big)\Big)$ and $DH\Big(\gamma\big(\hat{p}, \hat{u}_0, (\hat{\mathbf{f}}, \hat{\mathbf{g}})^\top\big)\Big)$ introduced in Assumption 116 and Assumption 118, respectively, we can consider the operator $DF\big(\hat{p}, \hat{u}_0, (\hat{\mathbf{f}}, \hat{\mathbf{g}})^\top\big)$ to be extended to the bounded linear operator

$$\begin{aligned} DF\big(\hat{p}, \hat{u}_0, (\hat{\mathbf{f}}, \hat{\mathbf{g}})^\top\big) : \quad & L^\infty(D, \mathbb{R}^{3+3L}) \times X \times L^1\big((0,t_1),\ L^2(D, \mathbb{R}^3 \times \mathbb{R}^{3\times3}_{\mathrm{sym}})\big) \\ &\quad \to \quad L^2\big(D \times (0,t_1), \mathbb{R}^3 \times \mathbb{R}^{3\times3}_{\mathrm{sym}}\big)\,. \end{aligned}$$

□

Lemma 124. *The adjoint of* (7.26) *is given by*

$$\begin{aligned} \pi_2^* : \quad L^2\big(D \times (0,t_1), \mathbb{R}^3 \times \mathbb{R}^{3\times 3}_{\mathrm{sym}}\big) \quad &\to \quad L^2\Big((0,t_1), \big(X, (\,.\,,\,.\,)_{T(\hat{P})}\big)\Big), \\ \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma} \end{pmatrix} \quad &\mapsto \quad \begin{pmatrix} \hat{\vartheta}\mathbf{v} \\ C\Big(\hat{\mu}_H + \sum_{l=1}^L \hat{\mu}_{M,l}\ ,\ \hat{\kappa}_H + \sum_{l=1}^L \hat{\kappa}_{M,l}\Big)\boldsymbol{\sigma} \\ -\hat{\omega}_{\boldsymbol{\sigma},1} C(\hat{\mu}_{M,1}, \hat{\kappa}_{M,1})\boldsymbol{\sigma} \\ \vdots \\ -\hat{\omega}_{\boldsymbol{\sigma},L} C(\hat{\mu}_{M,L}, \hat{\kappa}_{M,L})\boldsymbol{\sigma} \end{pmatrix}. \end{aligned} \tag{7.29}$$

The adjoint of (7.27) *is of the form*

$$\begin{aligned} \Big[\pi_2 T\big(\Gamma(\hat{p})\big)^{-1}\Big]^* : \quad L^2\big(D \times (0,t_1), \mathbb{R}^3 \times \mathbb{R}^{3\times 3}_{\mathrm{sym}}\big) \quad &\to \quad L^2\Big((0,t_1), \big(X, (\,.\,,\,.\,)_{E,\hat{P}}\big)\Big), \\ \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma} \end{pmatrix} \quad &\mapsto \quad \begin{pmatrix} \hat{\vartheta}\mathbf{v} \\ C(\hat{\mu}_H, \hat{\kappa}_H)\boldsymbol{\sigma} \\ C(\hat{\mu}_{M,1}, \hat{\kappa}_{M,1})\boldsymbol{\sigma} \\ \vdots \\ C(\hat{\mu}_{M,L}, \hat{\kappa}_{M,L})\boldsymbol{\sigma} \end{pmatrix}. \end{aligned} \tag{7.30}$$

And for the adjoint of (7.28), *we have*

$$\begin{aligned} D\gamma\big(\hat{p}, \hat{u}_0, (\hat{\mathbf{f}}, \hat{\mathbf{g}})^\top\big)^* : \quad & \mathcal{T}\,\Gamma(\mathcal{P})' \times X \times L^\infty\big((0,t_1), X\big) \\ & \to \quad L^\infty(D, \mathbb{R}^{3+3L})' \times X \times L^\infty\big((0,t_1),\ L^2(D, \mathbb{R}^3 \times \mathbb{R}^{3\times 3}_{\mathrm{sym}})\big), \\ \big(P', u_0, (\mathbf{f}, \mathbf{g}, \mathbf{h})^\top\big) \ \mapsto \ & \Big(\big(\vartheta' + \int_0^{t_1} \hat{\mathbf{f}}(t) \cdot \mathbf{f}(t)\, dt\ ,\ \mu_H', \dots, \omega_{\boldsymbol{\sigma},L}'\big), u_0, (\hat{\vartheta}\mathbf{f}, \mathbf{g})^\top\Big), \end{aligned} \tag{7.31}$$

where $(\vartheta', \mu_H', \dots, \omega_{\boldsymbol{\sigma},L}') \coloneqq D\Gamma^* P'$ *with the adjoint* $D\Gamma^*$ *of* $D\Gamma$, *which is an isometry, since* $D\Gamma$ *is an isometry by Lemma 102.*

Proof. These claims can be verified via direct calculations. To compute the first two of these three adjoints, we need the explicit form of the scalar products $(\,.\,,\,.\,)_{E,\hat{P}}$ from (5.32) and $(\,.\,,\,.\,)_{T(\hat{P})}$ from (5.33), which are repeated here for the convenience of the reader. It is

$$\begin{aligned} (w_1, w_2)_{E,\hat{P}} \quad = \quad & \Big(\frac{1}{\vartheta}\mathbf{v}^{(1)}\ ,\ \mathbf{v}^{(2)}\Big)_{L^2(D,\mathbb{R}^3)} \ + \ \Big(C\Big(\frac{1}{\mu_H}, \frac{1}{\kappa_H}\Big)\boldsymbol{\sigma}_H^{(1)}\ ,\ \boldsymbol{\sigma}_H^{(2)}\Big)_{L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}})} \\ & + \ \sum_{l=1}^L \Big(C\Big(\frac{1}{\mu_{M,l}}\ ,\ \frac{1}{\kappa_{M,l}}\Big)\boldsymbol{\sigma}_{M,l}^{(1)}\ ,\ \boldsymbol{\sigma}_{M,l}^{(2)}\Big)_{L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}})}, \end{aligned}$$

$w_i = \big(\mathbf{v}^{(i)}, \boldsymbol{\sigma}_H^{(i)}, \boldsymbol{\sigma}_M^{(i)}\big)^\top \in X, \quad i=1,2, \quad \text{and}$

$$
\begin{aligned}
&(u_1,u_2)_{T(\hat{P})}\\
&\quad = \Big(\frac{1}{\vartheta}\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\Big)_{L^2(D,\mathbb{R}^3)}\\
&\qquad + \bigg(C\Big(\frac{1}{\mu_H}, \frac{1}{\kappa_H}\Big)\Big(\boldsymbol{\sigma}^{(1)} + \sum_{l=1}^{L}\frac{1}{\omega_{\boldsymbol{\sigma},l}}\boldsymbol{\eta}_l^{(1)}\Big)\,,\ \boldsymbol{\sigma}^{(2)} + \sum_{l=1}^{L}\frac{1}{\omega_{\boldsymbol{\sigma},l}}\boldsymbol{\eta}_l^{(2)}\bigg)_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})}\\
&\qquad + \sum_{l=1}^{L}\bigg(\frac{1}{\omega_{\boldsymbol{\sigma},l}^2}C\Big(\frac{1}{\mu_{M,l}}\,,\ \frac{1}{\kappa_{M,l}}\Big)\boldsymbol{\eta}_l^{(1)}\,,\ \boldsymbol{\eta}_l^{(2)}\bigg)_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})}\,,
\end{aligned}
$$

$u_i = \big(\mathbf{v}^{(i)}, \boldsymbol{\sigma}^{(i)}, \boldsymbol{\eta}^{(i)}\big)^\top \in X, \quad i=1,2.$ □

The following two theorems constitute the main result of this section. The first one shows how the adjoint of $DF\big(\hat{p}, \hat{u}_0, (\hat{\mathbf{f}}, \hat{\mathbf{g}})^\top\big)$ for some point $\big(\hat{p}, \hat{u}_0, (\hat{\mathbf{f}}, \hat{\mathbf{g}})^\top\big)$ of the domain of F can be computed using the transformed variables $\mathbf{v}$, $\boldsymbol{\sigma}_H$, $\boldsymbol{\sigma}_M$. The second one derives an analogous result in terms of the original variables $\mathbf{v}$, $\boldsymbol{\sigma}$ and $\boldsymbol{\eta}$.

Theorem 125. *The adjoint*

$$
\begin{aligned}
DF\big(\hat{p}, \hat{u}_0, (\hat{\mathbf{f}}, \hat{\mathbf{g}})^\top\big)^* :\ & L^2\big(D\times(0,t_1), \mathbb{R}^3\times\mathbb{R}^{3\times3}_{\mathrm{sym}}\big)\\
&\to L^\infty(D,\mathbb{R}^{3+3L})' \times X \times L^\infty\big((0,t_1), L^2(D,\mathbb{R}^3\times\mathbb{R}^{3\times3}_{\mathrm{sym}})\big)
\end{aligned}
$$

of the derivative of the parameter-to-solution-map F at a point $\big(\hat{p}, \hat{u}_0, (\hat{\mathbf{f}}, \hat{\mathbf{g}})^\top\big) \in L^\infty(D,\mathbb{R}^{3+3L}) \times \mathcal{D}(\tilde{Q}) \times W^{1,1}\big((0,t_1), L^2(D,\mathbb{R}\times\mathbb{R}^{3\times3}_{\mathrm{sym}})\big)$, where $\hat{p} = (\hat{\vartheta},\ldots,\hat{\omega}_{\boldsymbol{\sigma},L})$ and $\hat{u}_0 = (\hat{\mathbf{v}}^{(0)}, \hat{\boldsymbol{\sigma}}^{(0)}, \hat{\boldsymbol{\eta}}^{(0)})^\top$, is of the form

$$
DF\big(\hat{p}, \hat{u}_0, (\hat{\mathbf{f}}, \hat{\mathbf{g}})^\top\big)^* \begin{pmatrix}\mathbf{v}\\ \boldsymbol{\sigma}\end{pmatrix} = \big((\vartheta',\ldots,\omega'_{\boldsymbol{\sigma},L}), (\mathbf{v}^{(0)}, \boldsymbol{\sigma}^{(0)}, \boldsymbol{\eta}^{(0)})^\top, (\mathbf{f}, \mathbf{g})^\top\big)
$$

with

$$
\begin{aligned}
\vartheta' &= \frac{1}{\hat{\vartheta}}\int_0^{t_1}\bigg(\operatorname{div}\Big(\hat{\boldsymbol{\sigma}}_H + \sum_{l=1}^{L}\hat{\boldsymbol{\sigma}}_{M,l}\Big) + \hat{\mathbf{f}}\bigg)\cdot\bar{\mathbf{v}}\,dt\,,\\
\mu'_H &= \int_0^{t_1} C\Big(\frac{1}{\hat{\mu}_H}, 0\Big)\varepsilon(\hat{\mathbf{v}}) : \bar{\boldsymbol{\sigma}}_H\,dt\,,\\
\kappa'_H &= \int_0^{t_1} C\Big(0, \frac{1}{\hat{\kappa}_H}\Big)\varepsilon(\hat{\mathbf{v}}) : \bar{\boldsymbol{\sigma}}_H\,dt\,,\\
\mu'_{M,l} &= \int_0^{t_1} C\Big(\frac{1}{\hat{\mu}_{M,l}}, 0\Big)\varepsilon(\hat{\mathbf{v}}) : \bar{\boldsymbol{\sigma}}_{M,l}\,dt\,, \qquad l=1,\ldots,L\,,
\end{aligned}
$$

$$
\begin{aligned}
\kappa'_{M,l} &= \int_0^{t_1} C\Big(0, \frac{1}{\hat\kappa_{M,l}}\Big)\varepsilon(\hat{\mathbf{v}}) : \bar{\boldsymbol\sigma}_{M,l}\,dt\,, \qquad l = 1,\dots,L\,,\\
\omega'_{\boldsymbol\sigma,l} &= \frac{1}{\hat\omega^2_{\boldsymbol\sigma,l}}\hat{\boldsymbol\eta}^{(0)}_l : \bigg(C\Big(\frac{1}{\hat\mu_{M,l}}, \frac{1}{\hat\kappa_{M,l}}\Big)\bar{\boldsymbol\sigma}_{M,l}(0) - C\Big(\frac{1}{\hat\mu_H}, \frac{1}{\hat\kappa_H}\Big)\bar{\boldsymbol\sigma}_H(0)\bigg)\\
&\qquad - \int_0^{t_1} C\Big(\frac{1}{\hat\mu_{M,l}}, \frac{1}{\hat\kappa_{M,l}}\Big)\hat{\boldsymbol\sigma}_{M,l} : \bar{\boldsymbol\sigma}_{M,l}\,dt\,, \qquad l = 1,\dots,L\,,\\
\mathbf{v}^{(0)} &= \frac{1}{\hat\vartheta}\bar{\mathbf{v}}(0)\,,\\
\boldsymbol\sigma^{(0)} &= C\Big(\frac{1}{\hat\mu_H}, \frac{1}{\hat\kappa_H}\Big)\bar{\boldsymbol\sigma}_H(0)\,,\\
\boldsymbol\eta^{(0)}_l &= \frac{1}{\hat\omega_{\boldsymbol\sigma,l}}\bigg(C\Big(\frac{1}{\hat\mu_H}, \frac{1}{\hat\kappa_H}\Big)\bar{\boldsymbol\sigma}_H(0) - C\Big(\frac{1}{\hat\mu_{M,l}}, \frac{1}{\hat\kappa_{M,l}}\Big)\bar{\boldsymbol\sigma}_{M,l}(0)\bigg)\,, \quad l = 1,\dots,L\,,\\
\mathbf{f} &= \bar{\mathbf{v}}\,,\\
\mathbf{g} &= C\Big(\frac{1}{\hat\mu_H}, \frac{1}{\hat\kappa_H}\Big)\bar{\boldsymbol\sigma}_H\,,
\end{aligned}
$$

where $(\bar{\mathbf{v}}, \bar{\boldsymbol\sigma}_H, \bar{\boldsymbol\sigma}_M)$ *satisfies*

$$
\begin{aligned}
\frac{d}{dt}\bar{\mathbf{v}}(t_1 - t) &= -\,\hat\vartheta\ \mathrm{div}\Big(\bar{\boldsymbol\sigma}_H + \sum_{l=1}^{L}\bar{\boldsymbol\sigma}_{M,l}\Big)(t_1 - t) \;+\; \hat\vartheta\mathbf{v}(t_1 - t)\,,\\
\frac{d}{dt}\bar{\boldsymbol\sigma}_H(t_1 - t) &= -\,C(\hat\mu_H, \hat\kappa_H)\,\varepsilon\big(\bar{\mathbf{v}}(t_1 - t)\big) \;+\; C(\hat\mu_H, \hat\kappa_H)\boldsymbol\sigma(t_1 - t)\,,\\
\frac{d}{dt}\bar{\boldsymbol\sigma}_{M,l}(t_1 - t) &= -\,C(\hat\mu_{M,l}, \hat\kappa_{M,l})\,\varepsilon\big(\bar{\mathbf{v}}(t_1 - t)\big) \;-\; \hat\omega_{\boldsymbol\sigma,l}\,\bar{\boldsymbol\sigma}_{M,l}(t_1 - t)\\
&\quad +\; C(\hat\mu_{M,l}, \hat\kappa_{M,l})\boldsymbol\sigma(t_1 - t)\,, \qquad l = 1,\dots,L\,,
\end{aligned}
\tag{7.32}
$$

$$
\bar{\mathbf{v}}(t_1) \;=\; \mathbf{0}\,, \qquad \bar{\boldsymbol\sigma}_H(t_1) \;=\; \mathbf{0}\,, \qquad \bar{\boldsymbol\sigma}_M(t_1) \;=\; \mathbf{0}\,,
$$

$$
\bar{\mathbf{v}}(t_1 - t)\big|_{\partial D_D} \;=\; \mathbf{0}\,, \qquad \mathbf{n}^\top\Big(\bar{\boldsymbol\sigma}_H + \sum_{l=1}^{L}\bar{\boldsymbol\sigma}_{M,l}\Big)(t_1 - t)\Big|_{\partial D_N} \;=\; \mathbf{0}\,,
$$

$t \in [0, t_1]$, *in the mild sense and* $(\hat{\mathbf{v}}, \hat{\boldsymbol\sigma}_H, \hat{\boldsymbol\sigma}_M)^\top$ *satisfies*

$$
\begin{aligned}
\hat{\mathbf{v}}'(t) &= \hat\vartheta\ \mathrm{div}\Big(\hat{\boldsymbol\sigma}_H + \sum_{l=1}^{L}\hat{\boldsymbol\sigma}_{M,l}\Big)(t) \;+\; \hat\vartheta\,\hat{\mathbf{f}}(t)\,,\\
\hat{\boldsymbol\sigma}'_H(t) &= C(\hat\mu_H, \hat\kappa_H)\,\varepsilon\big(\hat{\mathbf{v}}(t)\big) \;+\; \hat{\mathbf{g}}(t)\,,\\
\hat{\boldsymbol\sigma}'_{M,l}(t) &= C(\hat\mu_{M,l}, \hat\kappa_{M,l})\,\varepsilon\big(\hat{\mathbf{v}}(t)\big) \;-\; \hat\omega_{\boldsymbol\sigma,l}\,\hat{\boldsymbol\sigma}_{M,l}(t)\,, \qquad l = 1,\dots,L\,,
\end{aligned}
$$

$$
\begin{aligned}
\hat{\mathbf{v}}(0) &= \hat{\mathbf{v}}^{(0)}\,, & \hat{\boldsymbol{\sigma}}_H(0) &= \hat{\boldsymbol{\sigma}}^{(0)} + \sum_{l=1}^{L} \frac{1}{\hat{\omega}_{\boldsymbol{\sigma},l}} \hat{\boldsymbol{\eta}}_l^{(0)}\,, \\
& & \hat{\boldsymbol{\sigma}}_{M,l}(0) &= -\frac{1}{\hat{\omega}_{\boldsymbol{\sigma},l}} \hat{\boldsymbol{\eta}}_l^{(0)}\,, \qquad l = 1,\ldots,L\,, \\
\hat{\mathbf{v}}(t)\Big|_{\partial D_D} &= \mathbf{0}\,, & \mathbf{n}^\top\Big(\hat{\boldsymbol{\sigma}}_H + \sum_{l=1}^{L} \hat{\boldsymbol{\sigma}}_{M,l}\Big)(t)\Big|_{\partial D_N} &= \mathbf{0}\,,
\end{aligned}
$$

$t \in [0, t_1]$, in the classical sense.

Proof. It is

$$
\begin{aligned}
& DF\big(\hat{p}, \hat{u}_0, (\hat{\mathbf{f}}, \hat{\mathbf{g}})^\top\big)^* \\
&\quad = \quad D\gamma\big(\hat{p}, \hat{u}_0, (\hat{\mathbf{f}}, \hat{\mathbf{g}})^\top\big)^* D\tilde{G}\Big(\gamma\big(\hat{p}, \hat{u}_0, (\hat{\mathbf{f}}, \hat{\mathbf{g}})^\top\big)\Big)^* \Big[\pi_2 T\big(\Gamma(\hat{p})\big)^{-1}\Big]^*.
\end{aligned}
$$

To find $D\tilde{G}\big(\gamma(\hat{p}, \hat{u}_0, (\hat{\mathbf{f}}, \hat{\mathbf{g}})^\top\big)^* \big[\pi_2 T(\Gamma(\hat{p}))^{-1}\big]^*$, we use Theorem 117 and evaluate the expression

$$
\begin{aligned}
& \big[D\tilde{G}(\hat{P}, \hat{u}_0, \hat{f})^* \varphi\big](P, u_0, f) \\
&\quad = \quad \int_0^{t_1} \Big(-\beta(P)\hat{w}(t) + \big[DT(\hat{P})P\big]\hat{f}(t) + T(\hat{P})f(t)\ ,\ \bar{w}(t)\Big)_{E,\hat{P}}\, dt \\
&\qquad\quad + \Big(\big[DT(\hat{P})P\big]\hat{u}_0 + T(\hat{P})u_0\ ,\ \bar{w}(0)\Big)_{E,\hat{P}}
\end{aligned}
$$

for arbitrary $P = D\Gamma(\vartheta, \ldots, \omega_{\boldsymbol{\sigma},L}) \in \mathcal{T}\,\Gamma(\mathcal{P})$, $u_0 = (\mathbf{v}^{(0)}, \boldsymbol{\sigma}^{(0)}, \boldsymbol{\eta}^{(0)})^\top \in X$ and $f = (\mathbf{f}, \mathbf{g}, \mathbf{h})^\top \in L^1\big((0, t_1), X\big)$, and with $\hat{P} = \Gamma(\hat{\vartheta}, \ldots, \hat{\omega}_{\boldsymbol{\sigma},L})$, $\hat{f} = (\hat{\vartheta}\hat{\mathbf{f}}, \hat{\mathbf{g}}, \mathbf{0})^\top$, $\hat{u}_0 = (\hat{\mathbf{v}}^{(0)}, \hat{\boldsymbol{\sigma}}^{(0)}, \hat{\boldsymbol{\eta}}^{(0)})^\top$, $\varphi = \Big[\pi_2 T\big(\Gamma(\hat{\vartheta}, \ldots, \hat{\omega}_{\boldsymbol{\sigma},L})\big)^{-1}\Big]^* (\mathbf{v}, \boldsymbol{\sigma})^\top$ from (7.30), $\bar{w} = (\bar{\mathbf{v}}, \bar{\boldsymbol{\sigma}}_H, \bar{\boldsymbol{\sigma}}_M)^\top$ satisfying

$$
\begin{aligned}
\frac{d}{dt}\bar{w}(t_1 - t) &= -\beta(\hat{P})^* \bar{w}(t_1 - t) \ + \ \varphi(t_1 - t)\,, \qquad t \in [0, t_1]\,, \\
\bar{w}(t_1) &= 0
\end{aligned}
$$

in the mild sense, and $\hat{w} = (\hat{\mathbf{v}}, \hat{\boldsymbol{\sigma}}_H, \hat{\boldsymbol{\sigma}}_M)^\top$ satisfying

$$
\begin{aligned}
\hat{w}'(t) &= -\beta(\hat{P})\hat{w}(t) \ + \ T(\hat{P})\hat{f}(t)\,, \qquad t \in [0, t_1]\,, \\
\hat{w}(0) &= T(\hat{P})\hat{u}_0
\end{aligned}
$$

in the classical sense. Afterwards we concatenate (7.31) with the derived representation of $D\tilde{G}\big(\gamma(\hat{p}, \hat{u}_0, (\hat{\mathbf{f}}, \hat{\mathbf{g}})^\top\big)^* \big[\pi_2 T(\Gamma(\hat{p}))^{-1}\big]^*$.

It should be mentioned, that in the statement of the theorem, the symbols $\mathbf{v}^{(0)}$, $\boldsymbol{\sigma}^{(0)}$, $\boldsymbol{\eta}^{(0)}$, $\mathbf{f}$ and $\mathbf{g}$ have a different meaning than here in the proof.

In the sequel, we display the intermediate results of the first step.

$$
\begin{aligned}
&\int_0^{t_1} \big(-\beta(P)\hat{w}(t), \bar{w}(t)\big)_{E,\hat{P}}\,dt \\
&= \int_0^{t_1} \left(\begin{pmatrix} \vartheta \operatorname{div}\Big(\hat{\boldsymbol{\sigma}}_H + \sum_{l=1}^L \hat{\boldsymbol{\sigma}}_{M,l}\Big) \\ C\big(\mu_H, \kappa_H\big)\,\varepsilon(\hat{\mathbf{v}}) \\ C\big(\mu_{M,1}, \kappa_{M,1}\big)\,\varepsilon(\hat{\mathbf{v}}) - \omega_{\boldsymbol{\sigma},1}\hat{\boldsymbol{\sigma}}_{M,1} \\ \vdots \\ C\big(\mu_{M,L}, \kappa_{M,L}\big)\,\varepsilon(\hat{\mathbf{v}}) - \omega_{\boldsymbol{\sigma},L}\hat{\boldsymbol{\sigma}}_{M,L} \end{pmatrix}, \begin{pmatrix} \bar{\mathbf{v}} \\ \bar{\boldsymbol{\sigma}}_H \\ \bar{\boldsymbol{\sigma}}_{M,1} \\ \vdots \\ \bar{\boldsymbol{\sigma}}_{M,L} \end{pmatrix} \right)_{E,\hat{P}} dt \\
&= \int_0^{t_1} \left(\begin{pmatrix} \frac{\vartheta}{\hat{\vartheta}} \operatorname{div}\Big(\hat{\boldsymbol{\sigma}}_H + \sum_{l=1}^L \hat{\boldsymbol{\sigma}}_{M,l}\Big) \\ C\Big(\frac{\mu_H}{\hat{\mu}_H}, \frac{\kappa_H}{\hat{\kappa}_H}\Big)\,\varepsilon(\hat{\mathbf{v}}) \\ C\Big(\frac{\mu_{M,1}}{\hat{\mu}_{M,1}}, \frac{\kappa_{M,1}}{\hat{\kappa}_{M,1}}\Big)\,\varepsilon(\hat{\mathbf{v}}) - \omega_{\boldsymbol{\sigma},1} C\Big(\frac{1}{\hat{\mu}_{M,1}}, \frac{1}{\hat{\kappa}_{M,1}}\Big)\hat{\boldsymbol{\sigma}}_{M,1} \\ \vdots \\ C\Big(\frac{\mu_{M,L}}{\hat{\mu}_{M,L}}, \frac{\kappa_{M,L}}{\hat{\kappa}_{M,L}}\Big)\,\varepsilon(\hat{\mathbf{v}}) - \omega_{\boldsymbol{\sigma},L} C\Big(\frac{1}{\hat{\mu}_{M,L}}, \frac{1}{\hat{\kappa}_{M,L}}\Big)\hat{\boldsymbol{\sigma}}_{M,L} \end{pmatrix}, \begin{pmatrix} \bar{\mathbf{v}} \\ \bar{\boldsymbol{\sigma}}_H \\ \bar{\boldsymbol{\sigma}}_{M,1} \\ \vdots \\ \bar{\boldsymbol{\sigma}}_{M,L} \end{pmatrix} \right)_X dt \\
&= \int_0^{t_1}\int_D \frac{\vartheta}{\hat{\vartheta}} \operatorname{div}\Big(\hat{\boldsymbol{\sigma}}_H + \sum_{l=1}^L \hat{\boldsymbol{\sigma}}_{M,l}\Big)\cdot \bar{\mathbf{v}}\,dx\,dt \\
&\quad + \int_0^{t_1}\int_D C\Big(\frac{\mu_H}{\hat{\mu}_H}, \frac{\kappa_H}{\hat{\kappa}_H}\Big)\varepsilon(\hat{\mathbf{v}}) : \bar{\boldsymbol{\sigma}}_H\,dx\,dt \\
&\quad + \sum_{l=1}^L \int_0^{t_1}\int_D C\Big(\frac{\mu_{M,l}}{\hat{\mu}_{M,l}}, \frac{\kappa_{M,l}}{\hat{\kappa}_{M,l}}\Big)\varepsilon(\hat{\mathbf{v}}) : \bar{\boldsymbol{\sigma}}_{M,l}\,dx\,dt \\
&\quad - \sum_{l=1}^L \int_0^{t_1}\int_D \omega_{\boldsymbol{\sigma},l} C\Big(\frac{1}{\hat{\mu}_{M,l}}, \frac{1}{\hat{\kappa}_{M,l}}\Big)\hat{\boldsymbol{\sigma}}_{M,l} : \bar{\boldsymbol{\sigma}}_{M,l}\,dx\,dt \\
&= \int_D\int_0^{t_1} \frac{\vartheta}{\hat{\vartheta}} \operatorname{div}\Big(\hat{\boldsymbol{\sigma}}_H + \sum_{l=1}^L \hat{\boldsymbol{\sigma}}_{M,l}\Big)\cdot \bar{\mathbf{v}}\,dt\,dx \\
&\quad + \int_D\int_0^{t_1} \Big(\mu_H C\Big(\frac{1}{\hat{\mu}_H}, 0\Big) + \kappa_H C\Big(0, \frac{1}{\hat{\kappa}_H}\Big)\Big)\varepsilon(\hat{\mathbf{v}}) : \bar{\boldsymbol{\sigma}}_H\,dt\,dx \\
&\quad + \sum_{l=1}^L \int_D\int_0^{t_1} \Big(\mu_{M,l} C\Big(\frac{1}{\hat{\mu}_{M,l}}, 0\Big) + \kappa_{M,l} C\Big(0, \frac{1}{\hat{\kappa}_{M,l}}\Big)\Big)\varepsilon(\hat{\mathbf{v}}) : \bar{\boldsymbol{\sigma}}_{M,l}\,dt\,dx \\
&\quad - \sum_{l=1}^L \int_D\int_0^{t_1} \omega_{\boldsymbol{\sigma},l} C\Big(\frac{1}{\hat{\mu}_{M,l}}, \frac{1}{\hat{\kappa}_{M,l}}\Big)\hat{\boldsymbol{\sigma}}_{M,l} : \bar{\boldsymbol{\sigma}}_{M,l}\,dt\,dx
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \int_D \frac{\vartheta}{\hat{\vartheta}} \int_0^{t_1} \operatorname{div}\Big(\hat{\boldsymbol{\sigma}}_H + \sum_{l=1}^{L} \hat{\boldsymbol{\sigma}}_{M,l}\Big) \cdot \bar{\mathbf{v}}\, dt\, dx \\
& + \int_D \mu_H \int_0^{t_1} C\Big(\frac{1}{\hat{\mu}_H}, 0\Big)\varepsilon(\hat{\mathbf{v}}) : \bar{\boldsymbol{\sigma}}_H\, dt\, dx \\
& + \int_D \kappa_H \int_0^{t_1} C\Big(0, \frac{1}{\hat{\kappa}_H}\Big)\varepsilon(\hat{\mathbf{v}}) : \bar{\boldsymbol{\sigma}}_H\, dt\, dx \\
& + \sum_{l=1}^{L} \int_D \mu_{M,l} \int_0^{t_1} C\Big(\frac{1}{\hat{\mu}_{M,l}}, 0\Big)\varepsilon(\hat{\mathbf{v}}) : \bar{\boldsymbol{\sigma}}_{M,l}\, dt\, dx \\
& + \sum_{l=1}^{L} \int_D \kappa_{M,l} \int_0^{t_1} C\Big(0, \frac{1}{\hat{\kappa}_{M,l}}\Big)\varepsilon(\hat{\mathbf{v}}) : \bar{\boldsymbol{\sigma}}_{M,l}\, dt\, dx \\
& + \sum_{l=1}^{L} \int_D \omega_{\boldsymbol{\sigma},l}\Big(- \int_0^{t_1} C\Big(\frac{1}{\hat{\mu}_{M,l}}, \frac{1}{\hat{\kappa}_{M,l}}\Big)\hat{\boldsymbol{\sigma}}_{M,l} : \bar{\boldsymbol{\sigma}}_{M,l}\, dt\Big)\, dx\,.
\end{aligned}
$$

Furthermore, we recall that with $T(\hat{P})$ from (5.24), that is

$$
T(\hat{P}) \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma} \\ \boldsymbol{\eta}_1 \\ \vdots \\ \boldsymbol{\eta}_L \end{pmatrix} \quad = \quad \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma} + \sum_{l=1}^{L} \frac{1}{\hat{\omega}_{\boldsymbol{\sigma},l}} \boldsymbol{\eta}_l \\ -\frac{1}{\hat{\omega}_{\boldsymbol{\sigma},1}} \boldsymbol{\eta}_1 \\ \vdots \\ -\frac{1}{\hat{\omega}_{\boldsymbol{\sigma},L}} \boldsymbol{\eta}_L \end{pmatrix},
$$

and

$$
\big[DT(\hat{P})P\big] \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma} \\ \boldsymbol{\eta} \end{pmatrix} \quad = \quad \begin{pmatrix} \mathbf{0} \\ -\sum_{l=1}^{L} \frac{\omega_{\boldsymbol{\sigma},l}}{\hat{\omega}^2_{\boldsymbol{\sigma},l}} \boldsymbol{\eta}_l \\ \frac{\omega_{\boldsymbol{\sigma},1}}{\hat{\omega}^2_{\boldsymbol{\sigma},1}} \boldsymbol{\eta}_1 \\ \vdots \\ \frac{\omega_{\boldsymbol{\sigma},L}}{\hat{\omega}^2_{\boldsymbol{\sigma},L}} \boldsymbol{\eta}_L \end{pmatrix},
$$

from Lemma 106, it is

$$
\big[DT(\hat{P})P\big] \begin{pmatrix} \hat{\vartheta}\hat{\mathbf{f}} \\ \hat{\mathbf{g}} \\ \mathbf{0} \end{pmatrix} \;=\; 0\,, \qquad T(\hat{P}) \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \\ \mathbf{h} \end{pmatrix} \;=\; \begin{pmatrix} \mathbf{f} \\ \mathbf{g} + \sum_{l=1}^{L} \frac{1}{\hat{\omega}_{\boldsymbol{\sigma},l}} \mathbf{h}_l \\ -\frac{1}{\hat{\omega}_{\boldsymbol{\sigma},1}} \mathbf{h}_1 \\ \vdots \\ -\frac{1}{\hat{\omega}_{\boldsymbol{\sigma},L}} \mathbf{h}_L \end{pmatrix}
$$

and

$$\int_0^{t_1} \left(T(\hat{P}) \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \\ \mathbf{h} \end{pmatrix}, \begin{pmatrix} \bar{\mathbf{v}} \\ \bar{\boldsymbol{\sigma}}_H \\ \bar{\boldsymbol{\sigma}}_M \end{pmatrix} \right)_{E,\hat{P}} dt$$

$$= \int_0^{t_1} \int_D \frac{1}{\hat{\vartheta}} \bar{\mathbf{v}} \cdot \mathbf{f} \, dx \, dt \; + \; \int_0^{t_1} \int_D C\Big(\frac{1}{\hat{\mu}_H}, \frac{1}{\hat{\kappa}_H}\Big) \bar{\boldsymbol{\sigma}}_H : \mathbf{g} \, dx \, dt$$
$$+ \sum_{l=1}^{L} \int_0^{t_1} \int_D (\dots) : \mathbf{h}_l \, dx \, dt \,.$$

Finally,

$$\Big(\big[DT(\hat{P})P \big] \hat{u}_0 \;,\; \bar{w}(0) \Big)_{E,\hat{P}}$$
$$= \int_D \sum_{l=1}^{L} \frac{\omega_{\boldsymbol{\sigma},l}}{\hat{\omega}^2_{\boldsymbol{\sigma},l}} \hat{\boldsymbol{\eta}}_l^{(0)} : \left(C\Big(\frac{1}{\hat{\mu}_{M,l}}, \frac{1}{\hat{\kappa}_{M,l}}\Big) \bar{\boldsymbol{\sigma}}_{M,l}(0) - C\Big(\frac{1}{\hat{\mu}_H}, \frac{1}{\hat{\kappa}_H}\Big) \bar{\boldsymbol{\sigma}}_H(0) \right) dx$$

and

$$\Big(T(\hat{P}) u_0 \;,\; \bar{w}(0) \Big)_{E,\hat{P}}$$
$$= \int_D \frac{1}{\hat{\vartheta}} \bar{\mathbf{v}}(0) \cdot \mathbf{v}^{(0)} + C\Big(\frac{1}{\hat{\mu}_H}, \frac{1}{\hat{\kappa}_H}\Big) \bar{\boldsymbol{\sigma}}_H(0) : \boldsymbol{\sigma}^{(0)}$$
$$+ \sum_{l=1}^{L} \frac{1}{\hat{\omega}_{\boldsymbol{\sigma},l}} \left(C\Big(\frac{1}{\hat{\mu}_H}, \frac{1}{\hat{\kappa}_H}\Big) \bar{\boldsymbol{\sigma}}_H(0) - C\Big(\frac{1}{\hat{\mu}_{M,l}}, \frac{1}{\hat{\kappa}_{M,l}}\Big) \bar{\boldsymbol{\sigma}}_{M,l}(0) \right) : \boldsymbol{\eta}_l^{(0)} \; dx \,.$$

$\square$

How the adjoint of $DF\big(\hat{p}, \hat{u}_0, (\hat{\mathbf{f}}, \hat{\mathbf{g}})^\top\big)$ for some point $\big(\hat{p}, \hat{u}_0, (\hat{\mathbf{f}}, \hat{\mathbf{g}})^\top\big)$ of the domain of F can be computed using the original equation in the variables $\mathbf{v}$, $\boldsymbol{\sigma}$, $\boldsymbol{\eta}$, is shown in the following theorem.

Theorem 126. *The adjoint*

$$DF\big(\hat{p}, \hat{u}_0, (\hat{\mathbf{f}}, \hat{\mathbf{g}})^\top\big)^* : \; L^2\big(D \times (0,t_1), \mathbb{R}^3 \times \mathbb{R}^{3\times 3}_{\mathrm{sym}}\big)$$
$$\to \; L^\infty(D, \mathbb{R}^{3+3L})' \times X \times L^\infty\big((0,t_1), L^2(D, \mathbb{R}^3 \times \mathbb{R}^{3\times 3}_{\mathrm{sym}})\big)$$

of the derivative of the parameter-to-solution-map F at a point $\big(\hat{p}, \hat{u}_0, (\hat{\mathbf{f}}, \hat{\mathbf{g}})^\top\big) \in L^\infty(D, \mathbb{R}^{3+3L}) \times \mathcal{D}(\tilde{Q}) \times W^{1,1}\big((0,t_1), L^2(D, \mathbb{R} \times \mathbb{R}^{3\times 3}_{\mathrm{sym}})\big)$, where $\hat{p} = (\hat{\vartheta}, \dots, \hat{\omega}_{\boldsymbol{\sigma},L})$ and $\hat{u}_0 = (\hat{\mathbf{v}}^{(0)}, \hat{\boldsymbol{\sigma}}^{(0)}, \hat{\boldsymbol{\eta}}^{(0)})^\top$, is of the form

$$DF\big(\hat{p}, \hat{u}_0, (\hat{\mathbf{f}}, \hat{\mathbf{g}})^\top\big)^* \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma} \end{pmatrix} \;=\; \big((\vartheta', \dots, \omega'_{\boldsymbol{\sigma},L}), (\mathbf{v}^{(0)}, \boldsymbol{\sigma}^{(0)}, \boldsymbol{\eta}^{(0)})^\top, (\mathbf{f}, \mathbf{g})^\top \big)$$

with

$$\begin{aligned}
\vartheta' &= \frac{1}{\hat{\vartheta}}\int_0^{t_1} \big(\operatorname{div}\hat{\boldsymbol{\sigma}} + \hat{\mathbf{f}}\big)\cdot\bar{\mathbf{v}}\,dt\,,\\
\mu_H' &= \int_0^{t_1} C\Big(\frac{1}{\hat{\mu}_H},0\Big)\varepsilon(\hat{\mathbf{v}}) : \Big(\bar{\boldsymbol{\sigma}} + \sum_{l=1}^{L}\frac{1}{\hat{\omega}_{\boldsymbol{\sigma},l}}\bar{\boldsymbol{\eta}}_l\Big)\,dt\,,\\
\kappa_H' &= \int_0^{t_1} C\Big(0,\frac{1}{\hat{\kappa}_H}\Big)\varepsilon(\hat{\mathbf{v}}) : \Big(\bar{\boldsymbol{\sigma}} + \sum_{l=1}^{L}\frac{1}{\hat{\omega}_{\boldsymbol{\sigma},l}}\bar{\boldsymbol{\eta}}_l\Big)\,dt\,,\\
\mu_{M,l}' &= -\frac{1}{\hat{\omega}_{\boldsymbol{\sigma},l}}\int_0^{t_1} C\Big(\frac{1}{\hat{\mu}_{M,l}},0\Big)\varepsilon(\hat{\mathbf{v}}) : \bar{\boldsymbol{\eta}}_l\,dt\,, \qquad l=1,\dots,L\,,\\
\kappa_{M,l}' &= -\frac{1}{\hat{\omega}_{\boldsymbol{\sigma},l}}\int_0^{t_1} C\Big(0,\frac{1}{\hat{\kappa}_{M,l}}\Big)\varepsilon(\hat{\mathbf{v}}) : \bar{\boldsymbol{\eta}}_l\,dt\,, \qquad l=1,\dots,L\,,\\
\omega_{\boldsymbol{\sigma},l}' &= -\frac{1}{\hat{\omega}_{\boldsymbol{\sigma},l}}\int_0^{t_1} \big(C(\hat{\mu}_{M,l},\hat{\kappa}_{M,l})\varepsilon(\hat{\mathbf{v}}) + \hat{\boldsymbol{\eta}}_l\big)\\
&\qquad : \Big(\frac{1}{\hat{\omega}_{\boldsymbol{\sigma},l}}C\Big(\frac{1}{\hat{\mu}_{M,l}},\frac{1}{\hat{\kappa}_{M,l}}\Big)\bar{\boldsymbol{\eta}}_l\\
&\qquad\qquad + C\Big(\frac{1}{\hat{\mu}_H},\frac{1}{\hat{\kappa}_H}\Big)\Big(\bar{\boldsymbol{\sigma}} + \sum_{k=1}^{L}\frac{1}{\hat{\omega}_{\boldsymbol{\sigma},k}}\bar{\boldsymbol{\eta}}_k\Big)\Big)\,dt\,,\\
&\quad l=1,\dots,L\,,\\
\mathbf{v}^{(0)} &= \frac{1}{\hat{\vartheta}}\bar{\mathbf{v}}(0)\,,\\
\boldsymbol{\sigma}^{(0)} &= C\Big(\frac{1}{\hat{\mu}_H},\frac{1}{\hat{\kappa}_H}\Big)\Big(\bar{\boldsymbol{\sigma}}(0) + \sum_{l=1}^{L}\frac{1}{\hat{\omega}_{\boldsymbol{\sigma},l}}\bar{\boldsymbol{\eta}}_l(0)\Big)\,,\\
\boldsymbol{\eta}_l^{(0)} &= \frac{1}{\hat{\omega}_{\boldsymbol{\sigma},l}}\Big(C\Big(\frac{1}{\hat{\mu}_H},\frac{1}{\hat{\kappa}_H}\Big)\Big(\bar{\boldsymbol{\sigma}}(0) + \sum_{k=1}^{L}\frac{1}{\hat{\omega}_{\boldsymbol{\sigma},k}}\bar{\boldsymbol{\eta}}_k(0)\Big)\\
&\qquad + \frac{1}{\hat{\omega}_{\boldsymbol{\sigma},l}}C\Big(\frac{1}{\hat{\mu}_{M,l}},\frac{1}{\hat{\kappa}_{M,l}}\Big)\bar{\boldsymbol{\eta}}_l(0)\Big)\,,\\
&\quad l=1,\dots,L\,,\\
\mathbf{f} &= \bar{\mathbf{v}}\,,\\
\mathbf{g} &= C\Big(\frac{1}{\hat{\mu}_H},\frac{1}{\hat{\kappa}_H}\Big)\Big(\bar{\boldsymbol{\sigma}} + \sum_{l=1}^{L}\frac{1}{\hat{\omega}_{\boldsymbol{\sigma},l}}\bar{\boldsymbol{\eta}}_l\Big)\,,
\end{aligned}$$

where $(\bar{\mathbf{v}},\bar{\boldsymbol{\sigma}},\bar{\boldsymbol{\eta}})^\top$ *satisfies*

$$\frac{d}{dt}\bar{\mathbf{v}}(t_1-t) \;=\; -\hat{\vartheta}\operatorname{div}\bar{\boldsymbol{\sigma}}(t_1-t) \;+\; \hat{\vartheta}\mathbf{v}(t_1-t)\,,$$

$$\begin{aligned}
\frac{d}{dt}\bar{\boldsymbol{\sigma}}(t_1-t) \;&=\; -C\Big(\hat{\mu}_H+\sum_{l=1}^{L}\hat{\mu}_{M,l}\,,\;\hat{\kappa}_H+\sum_{l=1}^{L}\hat{\kappa}_{M,l}\Big)\,\varepsilon\big(\bar{\mathbf{v}}(t_1-t)\big)\;+\;\sum_{l=1}^{L}\bar{\boldsymbol{\eta}}_l(t_1-t)\\
&\quad+\;C\Big(\hat{\mu}_H+\sum_{l=1}^{L}\hat{\mu}_{M,l}\,,\;\hat{\kappa}_H+\sum_{l=1}^{L}\hat{\kappa}_{M,l}\Big)\boldsymbol{\sigma}(t_1-t)\,, \qquad (7.33)\\
\frac{d}{dt}\bar{\boldsymbol{\eta}}_l(t_1-t) \;&=\; \hat{\omega}_{\boldsymbol{\sigma},l}C(\hat{\mu}_{M,l}\,,\,\hat{\kappa}_{M,l})\,\varepsilon\big(\bar{\mathbf{v}}(t_1-t)\big)\;-\;\hat{\omega}_{\boldsymbol{\sigma},l}\bar{\boldsymbol{\eta}}_l(t_1-t)\\
&\quad-\;\hat{\omega}_{\boldsymbol{\sigma},l}C(\hat{\mu}_{M,l},\hat{\kappa}_{M,l})\boldsymbol{\sigma}(t_1-t)\,, \qquad l=1,\dots,L\,,
\end{aligned}$$

$$\begin{gathered}
\bar{\mathbf{v}}(t_1)\;=\;\mathbf{0}\,,\qquad \bar{\boldsymbol{\sigma}}(t_1)\;=\;\mathbf{0}\,,\qquad \bar{\boldsymbol{\eta}}(t_1)\;=\;\mathbf{0}\,,\\
\bar{\mathbf{v}}(t_1-t)\big|_{\partial D_D}\;=\;\mathbf{0}\,,\qquad \mathbf{n}^\top\bar{\boldsymbol{\sigma}}(t_1-t)\big|_{\partial D_N}\;=\;\mathbf{0}\,,
\end{gathered}$$

$t\in[0,t_1]$, in the mild sense and $(\hat{\mathbf{v}},\hat{\boldsymbol{\sigma}},\hat{\boldsymbol{\eta}})^\top$ satisfies

$$\begin{aligned}
\hat{\mathbf{v}}'(t) \quad&=\quad \hat{\vartheta}\,\mathrm{div}\,\hat{\boldsymbol{\sigma}}(t)\;+\;\hat{\vartheta}\hat{\mathbf{f}}(t)\,,\\
\hat{\boldsymbol{\sigma}}'(t) \quad&=\quad C\left(\hat{\mu}_H+\sum_{l=1}^{L}\hat{\mu}_{M,l}\,,\;\hat{\kappa}_H+\sum_{l=1}^{L}\hat{\kappa}_{M,l}\right)\varepsilon\big(\hat{\mathbf{v}}(t)\big)\;+\;\sum_{l=1}^{L}\hat{\boldsymbol{\eta}}_l(t)\;+\;\hat{\mathbf{g}}(t)\,,\\
\hat{\boldsymbol{\eta}}_l'(t) \quad&=\quad -\hat{\omega}_{\boldsymbol{\sigma},l}C(\hat{\mu}_{M,l}\,,\,\hat{\kappa}_{M,l})\,\varepsilon\big(\hat{\mathbf{v}}(t)\big)\;-\;\hat{\omega}_{\boldsymbol{\sigma},l}\hat{\boldsymbol{\eta}}_l(t)\,,\qquad l=1,\dots,L\,,
\end{aligned}$$

$$\begin{gathered}
\hat{\mathbf{v}}(0)\;=\;\hat{\mathbf{v}}^{(0)}\,,\qquad \hat{\boldsymbol{\sigma}}(0)\;=\;\hat{\boldsymbol{\sigma}}^{(0)}\,,\qquad \hat{\boldsymbol{\eta}}(0)\;=\;\hat{\boldsymbol{\eta}}^{(0)}\,,\\
\hat{\mathbf{v}}(t)\big|_{\partial D_D}\;=\;\mathbf{0}\,,\qquad \mathbf{n}^\top\hat{\boldsymbol{\sigma}}(t)\big|_{\partial D_N}\;=\;\mathbf{0}\,,
\end{gathered}$$

$t\in[0,t_1]$, in the classical sense.

Proof. It is

$$DF\big(\hat{p},\hat{u}_0,(\hat{\mathbf{f}},\hat{\mathbf{g}})^\top\big)^* \quad=\quad D\gamma\big(\hat{p},\hat{u}_0,(\hat{\mathbf{f}},\hat{\mathbf{g}})^\top\big)^*DH\Big(\gamma\big(\hat{p},\hat{u}_0,(\hat{\mathbf{f}},\hat{\mathbf{g}})^\top\big)\Big)^*\pi_2^*\,.$$

In a first step we derive a representation of $DH\big(\gamma(\hat{p},\hat{u}_0,(\hat{\mathbf{f}},\hat{\mathbf{g}})^\top)\big)^*\pi_2^*$ by using Theorem 119 and evaluating the expression

$$\begin{aligned}
&\big[DH(\hat{P},\hat{u}_0,\hat{f})^*\varphi\big](P,u_0,f)\\
&\quad=\;\int_0^{t_1}\big(-A\hat{u}(t)+f(t)\,,\;\bar{u}(t)\big)_{T(\hat{P})}\,dt\;+\;\big(u_0,\,\bar{u}(0)\big)_{T(\hat{P})}
\end{aligned} \qquad (7.34)$$

for arbitrary $P=D\Gamma(\vartheta,\dots,\omega_{\boldsymbol{\sigma},L})\in\mathcal{T}\,\Gamma(\mathcal{P})$, $u_0=(\mathbf{v}^{(0)},\boldsymbol{\sigma}^{(0)},\boldsymbol{\eta}^{(0)})^\top\in X$ and $f=(\mathbf{f},\mathbf{g},\mathbf{h})^\top\in L^1\big((0,t_1),X\big)$, and with $\hat{P}=\Gamma(\hat{\vartheta},\dots,\hat{\omega}_{\boldsymbol{\sigma},L})$, $\hat{f}=(\hat{\vartheta}\hat{\mathbf{f}},\hat{\mathbf{g}},\mathbf{0})^\top$, $\hat{u}_0=(\hat{\mathbf{v}}^{(0)},\hat{\boldsymbol{\sigma}}^{(0)},\hat{\boldsymbol{\eta}}^{(0)})^\top$, and $\varphi=\pi_2^*(\mathbf{v},\boldsymbol{\sigma})^\top$ from (7.29). The operator A has already been computed in the proof of proposition 112 on the derivative of F expressed in the original variables. It is

$$A\begin{pmatrix}\hat{\mathbf{v}}\\ \hat{\boldsymbol{\sigma}}\\ \hat{\boldsymbol{\eta}}\end{pmatrix} = \begin{pmatrix} -\vartheta\,\mathrm{div}\,\hat{\boldsymbol{\sigma}}\\ -C\Big(\mu_H+\sum_{l=1}^{L}\mu_{M,l}\,,\,\kappa_H+\sum_{l=1}^{L}\kappa_{M,l}\Big)\,\varepsilon(\hat{\mathbf{v}})\\ \hat{\omega}_{\boldsymbol{\sigma},1}C\big(\mu_{M,1},\kappa_{M,1}\big)\,\varepsilon(\hat{\mathbf{v}})+\omega_{\boldsymbol{\sigma},1}C\big(\hat{\mu}_{M,1},\hat{\kappa}_{M,1}\big)\,\varepsilon(\hat{\mathbf{v}})+\omega_{\boldsymbol{\sigma},1}\hat{\boldsymbol{\eta}}_1\\ \vdots\\ \hat{\omega}_{\boldsymbol{\sigma},L}C\big(\mu_{M,L},\kappa_{M,L}\big)\,\varepsilon(\hat{\mathbf{v}})+\omega_{\boldsymbol{\sigma},L}C\big(\hat{\mu}_{M,L},\hat{\kappa}_{M,L}\big)\,\varepsilon(\hat{\mathbf{v}})+\omega_{\boldsymbol{\sigma},L}\hat{\boldsymbol{\eta}}_L \end{pmatrix}.$$

Furthermore, $\hat{u}=(\hat{\mathbf{v}},\hat{\boldsymbol{\sigma}},\hat{\boldsymbol{\eta}})^\top$ satisfies

$$\begin{aligned} \hat{u}'(t) &= -\hat{A}\hat{u}(t) + \hat{f}(t)\,, \qquad t\in[0,t_1]\,,\\ \hat{u}(0) &= \hat{u}_0 \end{aligned}$$

in the classical sense with the original differential operator

$$\hat{A}\hat{u} = -\begin{pmatrix} \hat{\vartheta}\,\mathrm{div}\,\hat{\boldsymbol{\sigma}}\\ C\Big(\hat{\mu}_H+\sum_{l=1}^{L}\hat{\mu}_{M,l}\,,\,\hat{\kappa}_H+\sum_{l=1}^{L}\hat{\kappa}_{M,l}\Big)\,\varepsilon(\hat{\mathbf{v}}) + \sum_{l=1}^{L}\hat{\boldsymbol{\eta}}_l\\ -\hat{\omega}_{\boldsymbol{\sigma},1}C\big(\hat{\mu}_{M,1}\,,\,\hat{\kappa}_{M,1}\big)\,\varepsilon(\hat{\mathbf{v}}) - \hat{\omega}_{\boldsymbol{\sigma},1}\hat{\boldsymbol{\eta}}_1\\ \vdots\\ -\hat{\omega}_{\boldsymbol{\sigma},L}C\big(\hat{\mu}_{M,L}\,,\,\hat{\kappa}_{M,L}\big)\,\varepsilon(\hat{\mathbf{v}}) - \hat{\omega}_{\boldsymbol{\sigma},L}\hat{\boldsymbol{\eta}}_L \end{pmatrix}$$

containing the parameters $\hat{\vartheta},\ldots,\hat{\omega}_{\boldsymbol{\sigma},L}$, and $\bar{u}=(\bar{\mathbf{v}},\bar{\boldsymbol{\sigma}},\bar{\boldsymbol{\eta}})^\top$ satisfies

$$\begin{aligned} \frac{d}{dt}\bar{u}(t_1-t) &= -\hat{A}^*\bar{u}(t_1-t) + \varphi(t_1-t)\,, \qquad t\in[0,t_1]\,,\\ \bar{u}(t_1) &= 0 \end{aligned}$$

in the mild sense, where

$$\hat{A}^*\bar{u} = -\begin{pmatrix} -\hat{\vartheta}\,\mathrm{div}\,\bar{\boldsymbol{\sigma}}\\ -C\Big(\hat{\mu}_H+\sum_{l=1}^{L}\hat{\mu}_{M,l}\,,\,\hat{\kappa}_H+\sum_{l=1}^{L}\hat{\kappa}_{M,l}\Big)\,\varepsilon(\bar{\mathbf{v}}) + \sum_{l=1}^{L}\bar{\boldsymbol{\eta}}_l\\ \hat{\omega}_{\boldsymbol{\sigma},1}C\big(\hat{\mu}_{M,1}\,,\,\hat{\kappa}_{M,1}\big)\,\varepsilon(\bar{\mathbf{v}}) - \hat{\omega}_{\boldsymbol{\sigma},1}\bar{\boldsymbol{\eta}}_1\\ \vdots\\ \hat{\omega}_{\boldsymbol{\sigma},L}C\big(\hat{\mu}_{M,L}\,,\,\hat{\kappa}_{M,L}\big)\,\varepsilon(\bar{\mathbf{v}}) - \hat{\omega}_{\boldsymbol{\sigma},L}\bar{\boldsymbol{\eta}}_L \end{pmatrix}$$

has been computed in (7.25).

In a second step we concatenate (7.31) with the derived representation of $DH\big(\gamma(\hat{p},\hat{u}_0,(\hat{\mathbf{f}},\hat{\mathbf{g}})^\top)\big)^*\pi_2^*$.

Again we note that in the statement of the theorem, the symbols $\mathbf{v}^{(0)}$, $\boldsymbol{\sigma}^{(0)}$, $\boldsymbol{\eta}^{(0)}$, $\mathbf{f}$ and $\mathbf{g}$ have a different meaning than here in the proof.

In the sequel we display the intermediate results of the first step. We begin with

$$\int_0^{t_1} \big(- A\hat{u}(t)\,,\, \bar{u}(t)\big)_{T(\hat{P})}\,dt \quad = \quad \int_0^{t_1} \big(- T(\hat{P})A\hat{u}(t)\,,\, T(\hat{P})\bar{u}(t)\big)_{E,\hat{P}}\,dt\,. \quad (7.35)$$

It is

$$-T(\hat{P})A\hat{u} \quad = \quad \begin{pmatrix} \vartheta\,\mathrm{div}\,\hat{\boldsymbol{\sigma}} \\ C(\mu_H,\kappa_H)\varepsilon(\hat{\mathbf{v}}) - \sum_{l=1}^L \frac{\omega_{\boldsymbol{\sigma},l}}{\hat{\omega}_{\boldsymbol{\sigma},l}}\big(C(\hat{\mu}_{M,l},\hat{\kappa}_{M,l})\varepsilon(\hat{\mathbf{v}}) + \hat{\boldsymbol{\eta}}_l\big) \\ C(\mu_{M,1},\kappa_{M,1})\varepsilon(\hat{\mathbf{v}}) + \frac{\omega_{\boldsymbol{\sigma},1}}{\hat{\omega}_{\boldsymbol{\sigma},1}}\big(C(\hat{\mu}_{M,1},\hat{\kappa}_{M,1})\varepsilon(\hat{\mathbf{v}}) + \hat{\boldsymbol{\eta}}_1\big) \\ \vdots \\ C(\mu_{M,L},\kappa_{M,L})\varepsilon(\hat{\mathbf{v}}) + \frac{\omega_{\boldsymbol{\sigma},L}}{\hat{\omega}_{\boldsymbol{\sigma},L}}\big(C(\hat{\mu}_{M,L},\hat{\kappa}_{M,L})\varepsilon(\hat{\mathbf{v}}) + \hat{\boldsymbol{\eta}}_L\big) \end{pmatrix}.$$

So if we abbreviate $\tilde{w} := (\tilde{\mathbf{v}}, \tilde{\boldsymbol{\sigma}}_H, \tilde{\boldsymbol{\sigma}}_M)^\top := \hat{P}_1^{-1}T(\hat{P})\bar{u}$, the integrand of the iterated integral in (7.35) has the form

$$\vartheta\,\mathrm{div}\,\hat{\boldsymbol{\sigma}} : \tilde{\mathbf{v}} \;+\; C(\mu_H,\kappa_H)\varepsilon(\hat{\mathbf{v}}) : \tilde{\boldsymbol{\sigma}}_H \;+\; \sum_{l=1}^L C(\mu_{M,l},\kappa_{M,l})\varepsilon(\hat{\mathbf{v}}) : \tilde{\boldsymbol{\sigma}}_{M,l}$$

$$+\; \sum_{l=1}^L \frac{\omega_{\boldsymbol{\sigma},l}}{\hat{\omega}_{\boldsymbol{\sigma},l}}\big(C(\hat{\mu}_{M,l},\hat{\kappa}_{M,l})\varepsilon(\hat{\mathbf{v}}) + \hat{\boldsymbol{\eta}}_l\big) : (\tilde{\boldsymbol{\sigma}}_{M,l} - \tilde{\boldsymbol{\sigma}}_H)\,.$$

For the second term in (7.34) it holds

$$\begin{aligned} \int_0^{t_1} \big(f(t),\bar{u}(t)\big)_{T(\hat{P})}\,dt \quad &= \quad \int_0^{t_1} \left(T(\hat{P}) \begin{pmatrix} \mathbf{f}(t) \\ \mathbf{g}(t) \\ \mathbf{0} \end{pmatrix},\; \hat{P}_1^{-1}T(\hat{P})\bar{u}(t) \right)_X dt \\ &= \quad \int_0^{t_1} \left(\begin{pmatrix} \mathbf{f} \\ \mathbf{g} + \sum_{l=1}^L \frac{1}{\hat{\omega}_{\boldsymbol{\sigma},l}}\mathbf{h}_l \\ -\frac{1}{\hat{\omega}_{\boldsymbol{\sigma},1}}\mathbf{h}_1 \\ \vdots \\ -\frac{1}{\hat{\omega}_{\boldsymbol{\sigma},L}}\mathbf{h}_L \end{pmatrix},\; \begin{pmatrix} \tilde{\mathbf{v}}(t) \\ \tilde{\boldsymbol{\sigma}}_H(t) \\ \tilde{\boldsymbol{\sigma}}_M(t) \end{pmatrix} \right)_X dt \end{aligned}$$

with $\tilde{\mathbf{v}}$, $\tilde{\boldsymbol{\sigma}}_H$, $\tilde{\boldsymbol{\sigma}}_M$ as above.

Finally, the last term of (7.34) has the form

$$\begin{aligned} \big(u_0\,,\,\bar{u}(0)\big)_{T(\hat{P})} \quad &= \quad \big(T(\hat{P})u_0\,,\, P_1^{-1}T(\hat{P})\bar{u}(0)\big)_X \\ &= \quad \left(\begin{pmatrix} \mathbf{v}^{(0)} \\ \boldsymbol{\sigma}^{(0)} + \sum_{l=1}^L \frac{1}{\hat{\omega}_{\boldsymbol{\sigma},l}}\boldsymbol{\eta}_l^{(0)} \\ -\frac{1}{\hat{\omega}_{\boldsymbol{\sigma},1}}\boldsymbol{\eta}_1^{(0)} \\ \vdots \\ -\frac{1}{\hat{\omega}_{\boldsymbol{\sigma},L}}\boldsymbol{\eta}_L^{(0)} \end{pmatrix},\; \begin{pmatrix} \tilde{\mathbf{v}}(0) \\ \tilde{\boldsymbol{\sigma}}_H(0) \\ \tilde{\boldsymbol{\sigma}}_M(0) \end{pmatrix} \right)_X. \end{aligned}$$

□

Remark 127. *We remark again at this point, that up to the inhomogeneities, the adjoint backwards in time equations* (7.32) *and* (7.33) *only differ by some minus signs from the related wave equations, which describe the forward problem. Therefore for each pair of equations, only one numerical solver is needed.*

We achieved this by using the natural weighted energy scalar products $(.\,.)_{E,\hat{P}}$ *and* $(.\,.)_{T(\hat{P})}$ *at the parameter point, where we take the derivative. Yet we would like to point out explicitly here, that in both cases the adjoint of the derivative of the parameter-to-solution-map is still taken with respect to the canonical unweighted scalar product* $\int_0^{t_1} (.\,,.)_{L^2(D,\mathbb{R}^3\times\mathbb{R}^{3\times 3}_{\mathrm{sym}})}\, dt$ *in the codomain* $L^2(D \times (0,t_1),\ \mathbb{R}^3 \times \mathbb{R}^{3\times 3}_{\mathrm{sym}})$ *of* F. □

Appendix

Bochner Integrable Functions on the Line

This section is included to clarify the notion of integrability of Hilbert space valued functions which we base our argumentations on. Since the main focus of this thesis does not lie in this field, we only briefly sketch some definitions and further omit some basic facts which are needed in our calculations. More details can be found in [10], [9] and [8] for example.

Throughout this section let $\big(X, \|.\|\big)$ denote a Banch space.

Among the numerous notions of measurability around [3], we pick the following.

Definition A.1. *Let $\big(I, \mathcal{B} \cap I, \lambda^1|_{\mathcal{B}\cap I}\big)$ be the measure space consisting in an interval $I \subseteq \mathbb{R}$, the trace σ-algebra $\mathcal{B} \cap I$ of the Borel σ-algebra $\mathcal{B}$ on $\mathbb{R}$ and the one-dimensional Lebesgue-measure λ^1 on I. A function $I \to X$ of the form $\sum_{k=1}^{n} \alpha_k 1_{\Omega_k}$, with $n \in \mathbb{N}$, $\alpha_k \in X$, $\Omega_k \subseteq I$ Borel measurable and pairwise disjoint with $\lambda^1(\Omega_k) < \infty$ for $k = 1, \dots, n$ and $1_{\dots}$ denoting the respective characteristic function, is called a **simple function**. A function $f : I \to X$ is called **(strongly) measurable**, iff there is a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions such that $\|f(t) - f_n(t)\| \to 0$, $n \to \infty$ for almost every $t \in I$. A measurable function $f : I \to X$ is called **Bochner integrable**, iff there is a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$ such that $\int_I \big\|f_n(t) - f(t)\big\|\, dt \to 0$, $n \to \infty$. In this case its **Bochner integral** over any measurable set $\Omega \subseteq I$ is defined as $\int_\Omega f(t)\, dt := \lim_{n\to\infty} \int_\Omega f_n(t)\, dt$ and $\int_\Omega f_n(t)\, dt := \sum_{k=1}^{K_n} \alpha_k^{(n)} \lambda^1(\Omega \cap \Omega_k^{(n)})$, where we assume f_n to be of the form $f_n = \sum_{k=1}^{K_n} \alpha_k^{(n)} 1_{\Omega_k^{(n)}}$, $n \in \mathbb{N}$.*

The space of equivalence classes of Bochner integrable functions which are equal almost everywhere on I is denoted by $L^1(I, X)$. It is equipped with the norm $\|f\|_{L^1(I,X)} := \int_I \|f(t)\|\, dt$ for every f.

Inductively we define the spaces of weakly differentiable functions by setting

$W^{0,1}(I,X) := L^1(I,X)$ *and*

$$\begin{aligned} W^{k+1,1}(I,X) \quad &:= \Big\{ f \in L^1(I,X) : \\ &\exists t_0 \in I,\, u \in X,\, g \in W^{k,1}(I,X) : \quad f(t) = u + \int_{t_0}^t g(s)\, ds \quad \text{f.a.a. } t \in I \Big\} \end{aligned}$$

for $k \in \mathbb{N}_0$. *In this notation* g *is called the derivative of* f *and denoted by* f'. *Furthermore for* $k \in \mathbb{N}_0$ *and* $f \in W^{k,1}(I,X)$ *we denote* $f^{(0)} := f$ *and* $f^{(j+1)} := (f^{(j)})'$ *for* $j = 0, \ldots, k-1$. *On* $W^{k,1}(I,X)$ *we consider the norm* $\|f\|_{W^{k,1}(I,X)} := \sum_{j=0}^k \|f^{(j)}\|_{L^1(I,X)}$, $f \in W^{k,1}(I,X)$, $k \in \mathbb{N}_0$. $\square$

In the context of Definition A.1 the theorem of Pettis states that a function $f : I \to X$ is strongly measurable, if and only if it is weakly measurable and there is a separable subspace $Y \subseteq X$ such that $f(t) \in Y$ for almost all $t \in I$ (see [8] for example). In our application X is a separable Hilbert space either way.

Furthermore, with the help of the Lebesgue differentiation theorem for vector valued functions one can verify that the weak derivatives of an element of $W^{k,1}(I,X)$ are unique in $L^1(I,X)$.

Lemma A.2. *(Sobolev Embedding Theorem) Every* $f \in W^{1,1}\big((0,t_1),X\big)$ *contains a continuous representative* $\tilde{f}$, *and the embedding*

$$\begin{aligned} \iota \ : \quad W^{1,1}\big((0,t_1),X\big) \quad &\hookrightarrow \quad C\big([0,t_1],X\big) \\ f \quad &\mapsto \quad \tilde{f} \end{aligned}$$

is bounded with

$$\|\iota\| \quad \leq \quad c_{CW} \quad := \quad \max\Big\{\frac{1}{t_1},\, 1\Big\}. \tag{A.1}$$

Proof. Let $f \in W^{1,1}\big((0,t_1),X\big)$. By definition there is $t_0 \in (0,t_1)$, $u \in X$ and $g(= f') \in L^1\big((0,t_1),X\big)$ such that f can be represented by the function $\tilde{f}(t) := u + \int_{t_0}^t f'(s)\, ds$, $t \in (0,t_1)$. We extend $\tilde{f}$ to $[0,t_1]$ by arbitrarily extending f' to $[0,t_1]$. For $s,t \in (0,t_1)$ with $s \leq t$ it holds

$$\|\tilde{f}(t) - \tilde{f}(s)\| \quad = \quad \Big\| \int_s^t f'(r)\, dr \Big\| \quad \leq \quad \int_s^t \|f'(r)\|\, dr$$

which by the theorem of dominated convergence and the integrability of f' on $[0,t_1]$ tends to zero for $s \to t$ and $t \to s$. So $\tilde{f}$ is continuous.

Furthermore,

$$\tilde{f}(t) \quad = \quad u \ + \int_{t_0}^s f'(r)\, dr \ + \int_s^t f'(r)\, dr \quad = \quad \tilde{f}(s) \ + \int_s^t f'(r)\, dr$$

for every $s \in [0, t_1]$. So

$$\begin{aligned} t_1 \|\tilde{f}(t)\| &= \int_0^{t_1} \|\tilde{f}(t)\|\, ds \quad \leq \quad \int_0^{t_1} \|\tilde{f}(s)\| + \int_{\min\{s,t\}}^{\max\{s,t\}} \|f'(r)\|\, dr\, ds \\ &\leq \|f\|_{L^1((0,t_1),X)} \; + \; t_1 \|f'\|_{L^1((0,t_1),X)}\,, \end{aligned}$$

$t \in [0, t_1]$. It follows

$$\|\tilde{f}\|_{C([0,t_1],X)} \quad \leq \quad \max\left\{\frac{1}{t_1},\, 1\right\} \|f\|_{W^{1,1}((0,t_1),X)}\,.$$

□

Lemma A.3. *The space* $C_c^\infty\big((0,t_1),X\big)$ *is dense in* $L^p\big((0,t_1),X\big)$ *for* $1 \leq p < \infty$.

Proof. Let $f \in L^p\big((0,t_1),X\big)$. In a first step we prove, that f can be approximated by simple functions with respect to the norm $\|.\|_{L^p((0,t_1),X)}$. Since f is measurable, there is a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions converging to f pointwise almost everywhere. Also the sequence of simple functions

$$g_n(t) \;:=\; \begin{cases} f_n(t)\,, & \|f_n(t)\| \;\leq\; 2\|f(t)\|\,, \\ 0\,, & \text{otherwise}\,, \end{cases} \qquad t \in (0,t_1)\,,$$

converges to f pointwise almost everywhere and has the property, that $\|g_n(t)\| \leq 2\|f(t)\|$ for almost all $t \in (0,t_1)$ and all $n \in \mathbb{N}$. So $\|g_n(t) - f(t)\|^p \to 0$, $n \to \infty$ for almost all $t \in (0,t_1)$, and $\|g_n(t) - f(t)\|^p \leq \big(\|g_n(t)\| + \|f(t)\|\big)^p \leq 3^p \|f(t)\|^p$ almost everywhere, and $\int_0^{t_1} \|f(t)\|^p\, dt < \infty$. Hence, by the theorem of dominated convergence, $\int_0^{t_1} \|g_n(t) - f(t)\|^p\, dt \to 0$, $n \to \infty$.

In a second step we prove that any simple function can be approximated by smooth functions with respect to the norm $\|.\|_{L^p((0,t_1),X)}$. Therefore let $\Omega \subseteq (0,t_1)$ be some measurable set and let $\alpha \in X$. Due to [15], Corollary 3.5 there is a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $C_c^\infty\big((0,t_1),\mathbb{R}\big)$ with $\lim_{n\to\infty} \|\varphi_n - 1_\Omega\|_{L^p((0,t_1),\mathbb{R})} = 0$. Then $\varphi_n \alpha \in C_c^\infty\big((0,t_1),X\big)$, $n \in \mathbb{N}$, and

$$\begin{aligned} \lim_{n\to\infty} \|\varphi_n \alpha - 1_\Omega \alpha\|_{L^p((0,t_1),X)} &= \lim_{n\to\infty} \int_0^{t_1} \big\|\varphi_n(t)\alpha - 1_\Omega(t)\alpha\big\|^p\, dt \\ &= \lim_{n\to\infty} \|\alpha\|^p \int_0^{t_1} \big|\varphi_n(t) - 1_\Omega(t)\big|^p\, dt \\ &= 0\,. \end{aligned}$$

By taking the sum over functions $\varphi_n \alpha$ with this property we can approximate any simple function $\sum_{k=1}^K \alpha_k 1_{\Omega_k}$ in $L^p\big((0,t_1),X\big)$.

Finally, we complete the proof by combining step one and step two. □

Lemma A.4. *The space* $C^\infty\big([0,t_1],X\big)$ *is dense in* $W^{k,1}\big((0,t_1),X\big)$, $k \in \mathbb{N}_0$.

Proof. Let $k \in \mathbb{N}_0$, $f \in W^{k,1}\big((0,t_1),X\big)$ and $t_0 \in (0,t_1)$. Then it is $f^{(k)} \in L^1\big((0,t_1),X\big)$. Hence by Lemma A.3 there exists a sequence $(\varphi_{k,n})_{n\in\mathbb{N}}$ in the space $C_c^\infty\big((0,t_1),X\big)$ with $\lim_{n\to\infty} \|\varphi_{k,n} - f^{(k)}\|_{L^1((0,t_1),X)} = 0$. For $k=0$ there is nothing more to show. For $k>0$ we define $\varphi_{k-1,n}(t) := f^{(k-1)}(t_0) + \int_{t_0}^t \varphi_{k,n}(s)\,ds$. Then $\varphi_{k,n} = \varphi'_{k-1,n}$, $\varphi_{k-1,n} \in C^\infty\big([0,t_1],X\big)$, $n \in \mathbb{N}$, and

$$
\begin{aligned}
\int_{t_0}^{t_1} \|\varphi_{k-1,n}(t) - f^{(k-1)}(t)\|\,dt \quad &= \quad \int_{t_0}^{t_1} \Big\| \int_{t_0}^{t} \varphi_{k,n}(s) - f^{(k)}(s)\,ds \Big\|\,dt \\
&\leq \quad t_1 \int_{t_0}^{t_1} \big\|\varphi_{k,n}(s) - f^{(k)}(s)\big\|\,ds \\
&\to \quad 0\,, \qquad n \to \infty\,.
\end{aligned}
$$

In this way we proceed inductively and eventually define $\varphi_n := \varphi_{0,n}$, $n \in \mathbb{N}$. Thus $\varphi_n \in C^\infty\big([0,t_1],X\big)$, $n \in \mathbb{N}$, and

$$
\|\varphi_n - f\|_{W^{k,1}((0,t_1),X)} \;=\; \sum_{j=0}^{k} \|\varphi_{j,n} - f^{(j)}\|_{L^1((0,t_1),X)} \;\to\; 0\,, \qquad n \to \infty\,.
$$

$\square$

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